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PAR
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Résumé

Ce mémoire présente mes travaux de maîtrise portant sur la mécanique quantique bicomplexe. Les nombres bicomplexes constituent l'une des généralisations possibles des nombres complexes et ceux-ci possèdent une structure algébrique d'anneau commutatif. Initialement, le projet de maîtrise était d'utiliser les nombres bicomplexes afin de construire un espace analogue à l'espace de Hilbert en mécanique quantique standard. Cet *espace de Hilbert bicomplexe*, ainsi que les opérateurs agissant sur celui-ci, forment ce que l'on pourrait appeler une *mécanique quantique bicomplexe*, et l'idée était d'utiliser celle-ci afin de résoudre finalement les problèmes de l'oscillateur harmonique ainsi que de l'atome d'hydrogène bicomplexe. Cependant, plusieurs difficultés d'ordre mathématique se sont présentées en cours de route, dont la solution a permis d'écrire pas moins de deux articles sur l'algèbre linéaire ainsi que sur les espaces de Hilbert bicomplexes [1, 2], si bien que le problème de l'atome d'hydrogène fut abandonné.

Ce mémoire, par articles, présente donc la solution algébrique du problème de l'oscillateur harmonique quantique dans un espace de Hilbert bicomplexe [3] ainsi que de nombreux résultats et outils mathématiques qu'il a été nécessaire de développer pour résoudre le problème.

Raphaël Gervais Lavoie　　　　　　Louis Marchildon

Abstract

This thesis presents the work done during my master's degree on bicomplex quantum mechanics. Bicomplex numbers are one of the possible generalizations of complex numbers and they possess the algebraic structure of a commutating ring. The initial goal of this master's degree was to build a structure analogous to the Hilbert spaces in standard quantum mechanics, but standing on bicomplex numbers instead of complex ones. This *bicomplex Hilbert space*, and the operators acting on it, form what we can call a bicomplex quantum mechanics and we wanted to use it to solve the problems of the quantum harmonic oscillator and hydrogen atom. However, some mathematical difficulties arised on the way and forced us to abandon the hydrogen atom problem. The solution of these difficulties allowed us to write two papers on bicomplex Hilbert spaces and bicomplex linear algebra [1, 2].

This thesis, by articles, explores the algebraic solution to the harmonic oscillator problem in the framework of bicomplex quantum mechanics [3] as well as many results and mathematical tools that we developed during this investigation.

Avant-propos

Je tiens à remercier en premier lieu mon directeur de recherche, le professeur Louis Marchildon, pour son soutien, ses conseils plus que judicieux, sa confiance ainsi que d'avoir accepté de me diriger dans ce domaine. Je le remercie par dessus tout pour la rigueur scientifique qu'il a su me transmettre tout au long de mes études. J'aimerais également remercier mon codirecteur, le professeur Dominic Rochon, pour m'avoir initié aux bicomplexes ainsi que pour la motivation qu'il a su me donner tout au long de cette maîtrise. Je tiens également à remercier le professeur Adel Antippa pour sa vision unique de la physique ainsi que tous les professeurs et membres du Département de Physique de l'UQTR pour l'ambiance chaleureuse qui y règne.

D'un point de vue financier, j'aimerais remercier le FQRNT du Québec pour l'octroi d'une bourse de maîtrise ainsi que le CRSNG du Canada pour une bourse d'été. Je remercie également la fondation de l'UQTR ainsi que le professeur Rochon qui, en m'offrant un contrat d'embauche, m'a fait découvrir les nombres bicomplexes, qui sont le point central de ma maîtrise.

Finalement, d'un point de vue personnel, je tiens à remercier mes amis, et surtout ma famille, pour le support et l'appui qu'ils m'ont fournis du début à la fin de cette maîtrise. Un merci particulier à ma conjointe Audrey, pour la lecture attentive des diverses versions de ce document, mais surtout pour m'avoir soutenu lors des moments plus difficiles.

À mes parents, mon frère et ma conjointe

The path of discovery runs through
series of inferences which are
deeply veiled by the darkness
of instinctive guessing[.]
Hans Reichenbach [4, p. 67].

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Chapitre 1

Introduction

[...] even today the physicist more often has a kind of faith in the correctness of the new principles than a clear understanding of them.
Werner Heisenberg [5, préface].

En physique, comme dans bien d'autres domaines, la recherche sur de nouveaux terrains est rarement faite sans raison, ou du moins sans intuition. C'est pourquoi, dans le présent chapitre, certaines des motivations ayant mené à ce mémoire seront exposées. La structure du mémoire sera également brièvement présentée.

1.1 Motivations

Il est entendu que les motivations présentées dans cette section ne constituent en aucun cas une preuve justificative du mémoire, mais représentent plutôt les réflexions *a priori* ainsi qu'*a posteriori* de l'auteur face au présent sujet.

1.1.1 Systèmes de nombres

Toute théorie physique est formulée à l'aide d'une algèbre, c'est-à-dire un ensemble de règles, incluant une ou plusieurs lois de composition, agissant sur les éléments d'un ensemble. Puisqu'il existe, en mathématiques, une myriade d'algèbres différentes s'appuyant sur une grande variété d'ensembles, toute théorie physique se doit, normalement, de bien spécifier l'algèbre ainsi que l'ensemble (le plus souvent un système de nombres) utilisés pour la construction de ladite théorie.

Bien que ce point puisse sembler évident, ou même insignifiant pour certains, la tâche n'en est pas moins ardue. En effet, comme le souligne Penrose [6, p. 59], il ne semble pas possible de déduire le système de nombres sur lequel s'appuie une théorie de l'expérience;

> *In the development of mathematical ideas, one important initial driving force has always been to find mathematical structures that accurately mirror the behaviour of the physical world. But it is normally not possible to examine the physical world itself in such precise detail that appropriately clear-cut mathematical notions can be abstracted directly from it.*

En fait, Penrose va même plus loin en remettant en question la pertinence d'utiliser un système de nombres continu, comme celui des réels, alors que depuis le tout début de la théorie quantique, nous savons qu'au moins certains aspects de la nature possèdent un caractère purement discret.

Comme l'indiquait déjà Reichenbach il y a plus de 60 ans [4, p. 66], plus souvent qu'autrement c'est l'instinct du physicien qui ouvre la voie;

> *It was the instinct of the physicist which pointed the way. It is true that the men who did the work felt obliged to adduce logical reasons for the establishment of their assumptions ; and it seems plausible that this apparently logical line of thought was an important tool in the hands of those who were confronted by the task of transforming ingenious guesses into mathematical formulae.*

L'idée est que le système de nombres et l'algèbre utilisés pour construire une théorie sont essentiellement arbitraires. La justification du choix du système de nombres utilisé provient essentiellement du succès ou de l'échec de la théorie à modéliser de façon adéquate le monde qui nous entoure. En ce sens, l'idée de construire une théorie à partir des nombres bicomplexes semble, *a priori*, tout aussi justifiée que d'utiliser les nombres complexes en mécanique quantique standard ou encore les nombres réels en mécanique classique. Ce n'est qu'une fois la théorie construite, et par confrontation avec l'expérience, qu'il sera possible de juger de la pertinence ou non de ce choix.

1.1.2 Structures des théories physiques

Dans un tout autre ordre d'idées, l'un des plus grands défis – sinon le plus grand – de la physique théorique moderne concerne l'unification de la théorie de la relativité et de la mécanique quantique au sein d'une unique théorie cohérente. La majorité des tentatives en ce sens tendent à montrer que les deux théories sont fondamentalement incohérentes pour toutes sortes de raisons qu'il serait trop long de développer ici (le lecteur peut se référer à [6, 7, 8, 9, 10]).

Cela dit, un point important reste à noter. Il semble que la mécanique quantique nécessite (ou du moins soit simplifiée par) l'utilisation des nombres complexes [11, 12, 13]. À l'inverse, la théorie de la relativité semble se simplifier lorsqu'elle est formulée à l'aide de l'algèbre hyperbolique [6, 14, 15, 16, 17]. De ceci, on peut penser qu'une structure mathématique adéquate à l'unification de la relativité et de la mécanique quantique devrait également unifier les nombres complexes et hyperboliques. Les nombres bicomplexes constituent justement, comme nous allons le voir plus en détail au chapitre 2, la plus simple structure mathématique, de dimension quatre, unifiant les nombres complexes et les nombres hyperboliques.

1.1.3 Choix du problème

L'oscillateur harmonique est sans l'ombre d'un doute l'un des problèmes les plus importants de la physique. C'est probablement le problème le plus étudié et le plus utilisé d'entre tous. Il fait son apparition dans pratiquement toutes les branches de la physique. En mécanique classique, il se manifeste sous la forme du pendule simple et de systèmes masse-ressorts de faible amplitude connus depuis Hooke et Newton [18, p. 507–519]. En thermodynamique, l'oscillateur harmonique est à la base de la solution apportée par Planck au problème du rayonnement du corps noir [19, chap. 1]. En physique de l'état solide et de la matière condensée, il est régulièrement utilisé pour modéliser le couplage entre les différents atomes d'un cristal [20, chap. 4]. En électromagnétisme, le modèle de l'oscillateur harmonique est utilisé pour schématiser des circuits oscillants, des antennes rayonnantes [21, chap. 10], etc.

Considérant ces faits, il semble naturel d'explorer, de prime abord, la mécanique quantique bicomplexe à l'aide du problème le plus simple qui soit, l'oscillateur harmonique quantique bicomplexe. À la section 3.1, nous donnons quelques motivations supplémentaires spécifiques à l'oscillateur harmonique quantique.

1.1.4 Monopole des nombres complexes

Finalement, dans la littérature scientifique, on retrouve un nombre important de travaux portant sur toutes sortes de généralisations de la mécanique quantique à l'aide d'algèbre « non standard », [22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36]. Nul doute que l'un des articles instigateurs en ce sens est celui de Birkhoff et von Neumann [37] qui a mis en évidence la possible extension de la mécanique quantique au corps des quaternions, et ce, dès 1936.

Cependant, à notre connaissance, il n'existe aucune solution explicite du problème de l'oscillateur harmonique quantique formulé dans une algèbre différente de celle des nombres complexes (exception faite de la solution présentée dans ce mémoire) [1]. Nous croyons qu'à lui seul, ce dernier point justifie amplement l'investigation menée en [3].

1.2 Structure du mémoire

Le présent mémoire étant un mémoire par articles, sa structure sera plutôt différente de celle d'un mémoire traditionnel. La première partie du mémoire sera composée à la fois d'une introduction aux nombres bicomplexes plus substantielle que celle figurant dans les articles, ainsi que d'éléments considérés superflus dans une publication, mais néanmoins utiles à la compréhension. La seconde partie sera composée des articles à proprement parler.

De façon plus détaillée, le chapitre 2 sera essentiellement une introduction aux nombres bicomplexes ainsi qu'à l'arsenal mathématique utile à la compréhension du mémoire. L'idée de la généralisation des nombres réels y est abordée avec une insistance sur le processus de complexification. Ensuite, les nombres bicomplexes sont présentés en tant que tels et ceux-ci sont brièvement situés par rapport aux autres structures de nombres. Les concepts de corps et d'anneaux algébriques sont également énoncés, ainsi que les structures d'espaces vectoriels et de modules.

Le chapitre 3 introduit finalement les deux articles présentés comme exigence partielle du mémoire. Un résumé en français précèdera chacun des articles et il sera également question du lien entre les deux ainsi que de la contribution de l'étudiant.

1. Adler [26] présente le problème de l'oscillateur harmonique en mécanique quantique quaternionique, mais ne donne que l'équation différentielle sans, à notre connaissance, résoudre à proprement parler le problème.

Chapitre 2

Nombres bicomplexes

If science raises a question like the
anthropic question that cannot be
answered in terms of processes that
obey the laws of nature, it becomes
rational to invoke an outside
agency such as God.
Lee Smolin [9, p. 197].

Les nombres bicomplexes, bien que très peu connus, sont l'une des généralisations possibles des nombres complexes à la dimension quatre. En fait, les nombres bicomplexes peuvent être vus dans un sens plus large encore. Pour mettre ce point en évidence, nous expliquerons un peu plus en détail le processus menant aux nombres bicomplexes. Pour ce faire, retournons au corps des nombres réels.

2.1 Généralisation des nombres réels

Historiquement, la première généralisation des nombres réels ($\mathbb{R}$) fut les nombres complexes ($\mathbb{C}$), notés

$$\mathbb{C} := \left\{ z = a + b\mathbf{i} \;\middle|\; a, b \in \mathbb{R}, \quad \mathbf{i}^2 = -1 \right\}. \tag{2.1}$$

Ceux-ci se présentèrent sous la forme de solution « acceptable » à bon nombre d'équations polynomiales. Quoique réticents au début [38, chap. 4], les gens finirent par accepter les nombres complexes et les intégrèrent un peu partout. Aujourd'hui, bon nombre de théories physiques ou de solutions utilisent les nombres complexes, soit par obligation, soit comme raccourcis dans les calculs.

Cependant, en tentant de mieux comprendre la généralisation des réels aux complexes, on se rendit compte qu'il existait également deux autres façons de généraliser les nombres réels. Ces deux autres extensions des réels sont les nombres hyperboliques ($\mathbb{D}$) et les nombres duaux ou paraboliques, que nous noterons $\mathbb{P}$, bien qu'il ne semble pas exister de consensus quant au symbole pour les représenter. Les nombres hyperboliques et duaux sont définis comme

$$\mathbb{D} := \left\{ d = a + b\mathbf{j} \;\middle|\; a, b \in \mathbb{R}, \quad \mathbf{j}^2 = 1 \right\}, \tag{2.2}$$

$$\mathbb{P} := \left\{ p = a + b\varepsilon \;\middle|\; a, b \in \mathbb{R}, \quad \varepsilon^2 = 0 \right\}. \tag{2.3}$$

On dit de ε qu'il est un élément *nilpotent*, tandis que $\mathbf{j}$ est parfois noté h et représente la partie « hallucinatoire » [39]. Pour une démonstration plus formelle de l'existence des nombres duaux et paraboliques, le lecteur peut se référer à l'argument de Yaglom ou de Kantor et Solodonikov, repris par Hucks [40, sec. 2].

2.1.1 Liens avec la physique

S'il n'est nullement besoin de commenter l'utilisation des nombres complexes en physique, il est cependant utile de donner quelques applications des nombres hyperboliques et duaux. Bien que les nombres duaux soient les moins bien connus des trois, ils sont tout de même utilisés pour investiguer certains problèmes de la physique. Par exemple, Frydryszak propose un modèle d'oscillateur harmonique utilisant des éléments nilpotents [41, 42]. Il propose également d'utiliser les nombres duaux afin de représenter certains systèmes physiques, où la partie nilpotente modéliserait des fermions, alors que la partie réelle représenterait des bosons [42]. Dans ce cas, la signification physique de la propriété $\varepsilon^2 = 0$ ne serait autre que le principe d'exclusion de Pauli pour les fermions.

Pour ce qui est des nombres hyperboliques, Kocik [30] étudie, entre autres, les relations entre l'équation de Schrödinger et les processus de diffusion via la nature des nombres complexes et hyperboliques. L'équation de Schrödinger hyperbolique ainsi que les transformations de Fourrier hyperboliques sont également étudiées par Zheng et Xuegang [43]. En fait, une relation entre les nombres hyperboliques et la relativité restreinte a été notée par Fjelstad [39] il y a 25 ans déjà, et a été citée à de nombreuses reprises depuis [36, 40, 44, 45].

2.2 Complexification

Une des méthodes de généralisation des nombres utilise ce que l'on pourrait appeler une « complexification ». Ce processus consiste à remplacer tous les nombres réels par des nombres complexes. Pour qu'une complexification ait lieu, il faut également que la nouvelle unité imaginaire soit linéairement indépendante des autres qui peuvent être présentes dans l'équation. Considérons, par exemple, le nombre complexe

$$a + b\mathbf{i_1}, \tag{2.4}$$

avec $a, b \in \mathbb{R}$ et $\mathbf{i_1}$ une unité imaginaire telle que $\mathbf{i_1}^2 = -1$. Effectuons la transformation suivante

$$a \mapsto x_1 + x_2\mathbf{i_1}, \tag{2.5}$$

$$b \mapsto x_3 + x_4\mathbf{i_1}, \tag{2.6}$$

avec $x_1, x_2, x_3, x_4 \in \mathbb{R}$. Remplaçant (2.5) et (2.6) dans (2.4), nous obtenons

$$a + b\mathbf{i_1} \mapsto x_1 + x_2\mathbf{i_1} + x_3\mathbf{i_1} + x_4\mathbf{i_1}^2 = (x_1 - x_4) + (x_2 + x_3)\mathbf{i_1}. \tag{2.7}$$

Clairement, on trouve que $a = x_1 - x_4 \in \mathbb{R}$ et $b = x_2 + x_3 \in \mathbb{R}$. L'équation (2.7) montre bien que lorsque la transformation est effectuée sur une unité imaginaire ($\mathbf{i_1}$) qui n'est pas linéairement indépendante des autres unités imaginaires présentes, la forme algébrique de l'équation n'est pas affectée par la transformation. En effet, nous avons toujours un nombre complexe (en $\mathbf{i_1}$) du type $a + b\mathbf{i_1}$.

Par contre, effectuons la transformation

$$a \mapsto x_1 + x_3\mathbf{i_2}, \tag{2.8}$$

$$b \mapsto x_2 + x_4\mathbf{i_2}, \tag{2.9}$$

toujours avec $x_1, x_2, x_3, x_4 \in \mathbb{R}$, mais cette fois avec $\mathbf{i_2}$ une unité imaginaire ($\mathbf{i_2}^2 = -1$) linéairement indépendante de $\mathbf{i_1}$. Nous trouvons donc

$$a + b\mathbf{i_1} \mapsto x_1 + x_2\mathbf{i_1} + x_3\mathbf{i_2} + x_4\mathbf{i_1}\mathbf{i_2}. \tag{2.10}$$

Clairement, cette équation possède maintenant deux parties imaginaires, d'où le terme

« complexification ». Bien évidemment, le premier terme est un terme réel, mais examinons un peu mieux le dernier terme. Le comportement des unités imaginaires, hyperboliques ainsi que paraboliques, est essentiellement déterminé par le comportement de leur carré. Aussi longtemps que les unités imaginaires $\mathbf{i_1}$ et $\mathbf{i_2}$ commutent entre elles, c'est-à-dire $\mathbf{i_1 i_2} = \mathbf{i_2 i_1}$, on voit que

$$(\mathbf{i_1 i_2})^2 = \mathbf{i_1 i_2 i_1 i_2} = \mathbf{i_1}^2\mathbf{i_2}^2 = (-1)(-1) = 1. \tag{2.11}$$

De ceci, on conclut que $\mathbf{i_1 i_2}$ possède le même comportement qu'une unité hyperbolique, et on peut noter

$$\mathbf{j} := \mathbf{i_1 i_2}. \tag{2.12}$$

L'équation (2.12) fait également ressortir un point important. Si nous effectuons une conjugaison complexe (notée $^-$) de toutes les unités imaginaires, nous obtenons

$$\bar{\mathbf{j}} = \bar{\mathbf{i}}_\mathbf{1}\bar{\mathbf{i}}_\mathbf{2} = (-\mathbf{i_1})(-\mathbf{i_2}) = \mathbf{i_1 i_2} = \mathbf{j}. \tag{2.13}$$

Donc, par construction, nous obtenons clairement $\bar{\mathbf{j}} = \mathbf{j}$. Ce dernier point est important puisque dans la majorité de la littérature, il est généralement admis que $\bar{\mathbf{j}} = -\mathbf{j}$, ce que nous n'obtenons pas. Par contre, comme nous pouvons le voir dans [34, 35, 46], il est également possible de définir deux autres types de conjugaison sur les bicomplexes, qui font en sorte que le conjugé de $\mathbf{j}$ soit $-\mathbf{j}$. En fait, les deux autres conjugaisons possibles sont simplement la conjugaison de $\mathbf{i_1}$ uniquement, puis de $\mathbf{i_2}$ uniquement. On peut cependant noter que la conjugaison sur les bicomplexes qui sera utilisée dans ce qui suit ainsi que dans les articles est celle représentée par (2.13).

2.3 Nombres bicomplexes

L'équation (2.10) forme ce que l'on appelle l'ensemble $\mathbb{T}$ des nombres bicomplexes

$$\mathbb{T} := \{w = x_1 + x_2\mathbf{i_1} + x_3\mathbf{i_2} + x_4\mathbf{j} \mid x_1, x_2, x_3, x_4 \in \mathbb{R}\}\,, \tag{2.14}$$

et

$$\mathbf{i_1 i_2} = \mathbf{j} = \mathbf{i_2 i_1}, \qquad \mathbf{i_1 j} = -\mathbf{i_2} = \mathbf{j i_1}, \qquad \mathbf{i_2 j} = -\mathbf{i_1} = \mathbf{j i_2}. \tag{2.15}$$

De (2.14), on voit clairement que les nombres bicomplexes contiennent à la fois les nombres complexes et les nombres hyperboliques. Il est intéressant de noter, comme le suggère la figure 2.1, que la complexification des nombres complexes, de même que la complexification des nombres hyperboliques, engendrent toutes deux la structure des nombres bicomplexes.

Le processus de complexification peut être appliqué de façon itérative. Lorsqu'on l'applique deux et trois fois, on génère respectivement les bicomplexes et les tricomplexes. En général, on génère les multicomplexes d'ordre n en appliquant n fois le processus de complexification. Il est entendu que $n = 0$ représente les nombres réels. La dimension des nombres multicomplexes, c'est-à-dire le nombre d'éléments linéairement indépendants formant le multicomplexe, est donnée par $\mathcal{D} = 2^n$.

Il est intéressant de noter, bien que ceci s'éloigne quelque peu de l'objectif de ce mémoire, qu'une généralisation des réels pourrait également être faite par « hyperbolisation ». Dans ce cas, l'hyperbolisation des complexes engendre encore une fois les bicomplexes, alors qu'une hyperbolisation des nombres hyperboliques mènerait à une nouvelle branche de nombres, que nous pourrions appeler les *multiperboliques*. Dans ce cas, une hyperbolisation d'ordre deux engendrerait les *biperboliques*, et ceux-ci seraient

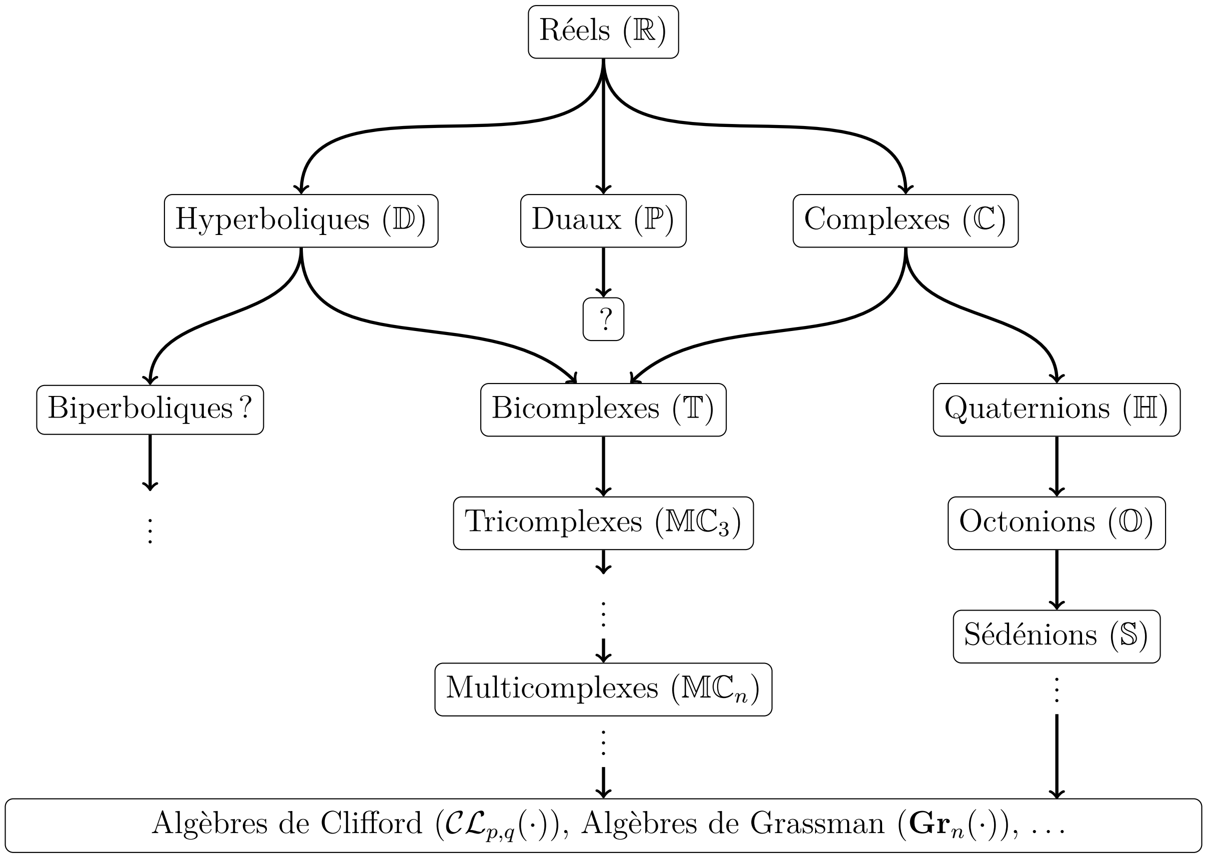

FIGURE 2.1 – Généralisation des nombres

donnés par

$$\text{biperboliques} := \{x_1 + x_2\mathbf{j} + x_3\mathbf{j_2} + x_4\mathbf{jj_2} \mid x_1, x_2, x_3, x_4 \in \mathbb{R}\}. \tag{2.16}$$

Dans ce cas, l'élément $\mathbf{jj_2}$ possède les propriétés d'une unité hyperbolique. Le processus d'hyperbolisation peut être poursuivi au même titre que celui de complexification.

2.3.1 Nombres bicomplexes *versus* quaternions

Bien que les nombres bicomplexes soient une généralisation des nombres réels de dimension quatre, ceux-ci forment néanmoins une structure algébrique complètement différente de l'autre généralisation des réels de dimension quatre bien connue, c'est-à-dire les quaternions. En effet, les quaternions forment un corps non commutatif, tandis que les nombres bicomplexes forment un anneau commutatif. Plus précisément, le corps des quaternions est défini comme

$$\mathbb{H} := \{x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k} \mid x_1, x_2, x_3, x_4 \in \mathbb{R}\}, \tag{2.17}$$

où la multiplication des unités 1, **i**, **j** et **k** est régie par [38, p. 295]

·	1	**i**	**j**	**k**
1	1	**i**	**j**	**k**
i	**i**	−1	**k**	−**j**
j	**j**	−**k**	−1	**i**
k	**k**	**j**	−**i**	−1

, $$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1. \tag{2.18}$$

La commutativité des bicomplexes implique que $ab = ba$ pour a, b deux scalaires bicomplexes (voir (2.15)), tandis que $ab \neq ba$ en général si a, b sont deux scalaires quaternioniques (voir (2.18)).

Le fait que les nombres bicomplexes forment un anneau algébrique plutôt qu'un corps provient du fait que les nombres bicomplexes comportent des éléments non inversibles, que l'on appelle *diviseurs de zéro*. En fait, a, b sont des diviseurs de zéro si

$$ab = 0, \qquad \text{et} \qquad a \neq 0 \neq b. \tag{2.19}$$

Si l'on définit

$$\mathbf{e_1} := \frac{1+\mathbf{j}}{2}, \qquad \mathbf{e_2} := \frac{1-\mathbf{j}}{2}, \tag{2.20}$$

on voit aisément que $\mathbf{e_1}\mathbf{e_2} = 0$. On a donc un exemple de diviseurs de zéros.

Les éléments $\mathbf{e_1}$ et $\mathbf{e_2}$ simplifient considérablement l'algèbre des nombres bicomplexes, comme nous le verrons en détail dans les articles.

2.4 Modules bicomplexes

Dans cette section, nous ferons un bref retour sur deux structures algébriques importantes, afin de mieux comprendre où se situent les nombres bicomplexes ainsi que les structures basées sur ceux-ci.

2.4.1 Corps et anneaux algébriques

Les concepts de *corps* et *d'anneaux* algébriques sont définis à partir d'un ensemble d'éléments et d'un certain nombre de lois de composition agissant sur les éléments de l'ensemble.

Anneau algébrique : Soit $S = \{x_i\}$, un ensemble et soit $+$ et $\cdot$ deux opérations binaires appelées *addition* et *multiplication* respectivement. L'ensemble S sera un anneau algébrique si et seulement si les propriétés suivantes sont respectées pour tout $x_i, x_j, x_k \in S$ [47, 48] ;

1. Fermeture sur l'addition : $x_i + x_j = x_k$ et $x_k \in S$,
2. Associativité de l'addition : $(x_i + x_j) + x_k = x_i + (x_j + x_k)$,
3. Neutre additif : $0 + x_i = x_i + 0 = x_i$,
4. Commutativité de l'addition : $x_i + x_j = x_j + x_i$,
5. Inverse additif : il existe un $x_j \in S$ tel que $x_i + x_j = 0$,
6. Fermeture sur la multiplication : $x_i \cdot x_j = x_k$ et $x_k \in S$,
7. Associativité de la multiplication : $(x_i \cdot x_j) \cdot x_k = x_i \cdot (x_j \cdot x_k)$,

8. Distributivité de la multiplication sur l'addition : $x_i \cdot (x_j + x_k) = x_i \cdot x_j + x_i \cdot x_k$ et $(x_i + x_j) \cdot x_k = x_i \cdot x_k + x_j \cdot x_k$.

Corps algébrique : Soit $S = \{x_i\}$, un ensemble et soit $+$ et $\cdot$ deux opérations binaires appelées *addition* et *multiplication* respectivement. L'ensemble S sera un corps algébrique si et seulement si l'ensemble S est un anneau algébrique, et possède en plus les propriétés

9. Neutre multiplicatif : $1 \cdot x_i = x_i \cdot 1 = x_i$,
10. Inverse multiplicatif : pour tout $x_i \in S$, il existe un $x_j \in S$ tel que $x_i \cdot x_j = 1$.

De ces deux définitions, on voit clairement qu'un anneau possède une structure plus générale qu'un corps. En d'autres mots, un corps est un cas particulier d'anneau algébrique. Le symbole $\cdot$ est normalement omis.

Un exemple bien simple de corps est celui des nombres réels. En effet, l'ensemble des réels satisfait aux dix caractéristiques d'un corps, plus quelques caractéristiques supplémentaires comme par exemple la commutativité sur la multiplication ($x \cdot y = y \cdot x,\ \ x, y \in \mathbb{R}$).

Les nombres bicomplexes, quant à eux, forment un anneau plutôt qu'un corps. En effet, la propriété 10 pose problème pour l'ensemble des bicomplexes. Par exemple, le fait que $\mathbf{e_1}\mathbf{e_2} = 0$ implique que ni $\mathbf{e_1}$, ni $\mathbf{e_2}$ ne possède d'inverse multiplicatif. Il n'est donc pas possible de trouver un $x \in \mathbb{T}$ tel que $\mathbf{e_k} \cdot x = 1$, $k = 1, 2$.

2.4.2 Espace vectoriel et module

Soit $\mathcal{V}$ un ensemble et S un corps, et soit $+$ et $\cdot$ deux opérations binaires. Supposons que pour tout $|u\rangle$, $|v\rangle$ et $|w\rangle$ dans $\mathcal{V}$, et pour tout α, β dans S, nous ayons

1. $|u\rangle + |v\rangle = |v\rangle + |u\rangle$,
2. $(|u\rangle + |v\rangle) + |w\rangle = |u\rangle + (|v\rangle + |w\rangle)$,
3. il existe un élément $|0\rangle$ dans $\mathcal{V}$ tel que $|0\rangle + |u\rangle = |u\rangle$,
4. $0 \cdot |u\rangle = |0\rangle$,
5. $1 \cdot |u\rangle = |u\rangle$,
6. $\alpha \cdot (|u\rangle + |v\rangle) = \alpha \cdot |u\rangle + \alpha \cdot |v\rangle$,
7. $(\alpha + \beta) \cdot |u\rangle = \alpha \cdot |u\rangle + \beta \cdot |u\rangle$,
8. $(\alpha\beta) \cdot |u\rangle = \alpha \cdot (\beta \cdot |u\rangle)$,

alors le quadruplet $(\mathcal{V}, S, +, \cdot)$ est appelé *espace vectoriel* [49, 50].

Les espaces vectoriels sont d'une importance capitale puisque ce sont généralement dans ces espaces que vivent tous les vecteurs d'une théorie, et ce, que ce soit des vecteurs de l'espace des phases ou bien de l'espace physique à une, deux, trois ou encore quatre dimensions.

Un *module* possède les mêmes propriétés qu'un espace vectoriel, mais repose sur un anneau algébrique plutôt qu'un corps. De ceci, on déduit directement que la structure analogue à celle d'espace vectoriel, mais reposant sur l'anneau des bicomplexes, forme un module. Nous noterons $\mathbb{T}$-*module* un module construit sur les nombres bicomplexes.

Chapitre 3

Présentation des articles

Our imagination is stretched to the
utmost, not, as in fiction, to imagine
things which are not really there,
but just to comprehend those
things which are *there.*
Richard P. Feynman [51, p. 121].

3.1 Premier article

Le problème de l'oscillateur harmonique est l'un des problèmes les plus importants et les plus étudiés en mécanique quantique. Pour ne donner qu'un exemple, il suffit de regarder le titre du chapitre 2 de Böhm [52] : *Foundations of Quantum Mechanics – The Harmonic Oscillator*. L'importance de ce problème provient fort probablement de deux points majeurs. Premièrement, l'oscillateur harmonique est le problème non trivial le plus simple. Deuxièmement, l'oscillateur harmonique quantique possède une solution analytique. Ce dernier point peut sembler évident, mais rappelons qu'en mécanique quantique, peu nombreux sont les problèmes ayant une solution analytique. Lorsqu'il n'existe pas de solutions analytiques, il faut recourir aux solutions numériques, ou encore aux méthodes de perturbations utilisant normalement un problème connu comme point de départ. En ce sens, l'une des grandes utilités de l'oscillateur harmonique quantique est justement de tenir lieu de problème connu dans la théorie des perturbations.

3.1.1 Solution différentielle

Non seulement l'oscillateur harmonique quantique se résout de façon analytique, mais il existe en fait deux façons différentes de résoudre le problème. La première méthode consiste à transformer l'équation aux valeurs propres de l'Hamiltonien de l'oscillateur harmonique en une équation différentielle, puis de la résoudre. Pour ce faire, écrivons l'Hamiltonien de l'oscillateur harmonique quantique,

$$H = \frac{P^2}{2m} + \frac{1}{2} m\omega^2 X^2, \tag{3.1}$$

où P est l'opérateur d'impulsion et X l'opérateur de position. Les quantités m et ω sont deux nombres réels positifs normalement associés à la masse et à la fréquence de l'oscillateur. À noter que dans la définition de (3.1), H, X et P sont des opérateurs hermitiques. L'Hamiltonien à lui seul ne suffit cependant pas à spécifier totalement le problème. Il faut en plus définir dans quel espace agissent les opérateurs H, X et P. Traditionnellement, ceux-ci agissent dans un espace vectoriel complexe de dimension infinie. Dans notre cas, pour l'oscillateur bicomplexe, notre « espace » sera un module ($\mathbb{T}$-module) bicomplexe de dimension infinie.

Pour transformer l'équation aux valeurs propres associée à (3.1) en une équation différentielle, nous devons spécifier l'action des opérateurs H, X et P sur les éléments de l'ensemble. Pour ce faire, il est nécessaire d'introduire un postulat supplémentaire, soit sur l'action de l'opérateur P, soit sur la relation de commutation de X et P. Dans notre cas, nous avons choisi la seconde option.

Postulat 1. *La relation de commutation des opérateurs X et P, agissant dans un module bicomplexe M, est un multiple de l'identité.*

En d'autres mots, le postulat 1 peut s'écrire comme

$$[X, P] = wI, \tag{3.2}$$

avec $w \in \mathbb{T}$, une constante bicomplexe et I l'opérateur identité sur le module bicomplexe M. Sans perte de généralité, nous pouvons écrire

$$[X, P] = \mathbf{i_1}\hbar\xi I, \qquad \xi \in \mathbb{T}. \tag{3.3}$$

Il n'est pas tellement difficile de montrer [3] que la constante ξ doit être hyperbolique pour que X et P soient des opérateurs hermitiques. De plus, si l'on demande que les valeurs propres de l'opérateur de position (X) soient des nombres réels et que l'on utilise les propriétés du delta de Dirac, on peut montrer que l'action de X et P

est donnée par

$$X\{u(x)\} \mapsto xu(x), \qquad P\{u(x)\} \mapsto -\mathbf{i_1}\hbar\xi\frac{d}{dx}u(x), \tag{3.4}$$

où $u(x)$ sont les fonctions propres de l'Hamiltonien, telles que

$$H\{u(x)\} = Eu(x). \tag{3.5}$$

En appliquant l'Hamiltonien (3.1) sur une fonction propre, et en utilisant (3.4), on obtient l'équation différentielle

$$-\frac{\hbar^2\xi^2}{2m}\frac{d^2}{dx^2}u(x) + \frac{1}{2}m\omega^2x^2u(x) = Eu(x). \tag{3.6}$$

Il suffit alors de résoudre cette équation pour trouver les valeurs propres ainsi que les fonctions propres de l'oscillateur harmonique quantique bicomplexe. À noter cependant que (3.6) est une équation différentielle bicomplexe à variable réelle au sens ou ξ et E sont des constantes hyperboliques (du fait que H est hermitique) et $u(x)$ est une fonction bicomplexe d'une variable réelle. Cependant, pour des raisons de symétrie, on peut montrer que $u(x)$ est en fait une fonction hyperbolique d'une variable réelle.

3.1.2 Solution algébrique

Dans ce mémoire, nous utiliserons cependant une approche différente, en l'occurence la solution algébrique. L'approche algébrique du problème de l'oscillateur harmonique quantique, quoique plus longue que l'approche différentielle, permet de mieux comprendre le comportement des solutions. En effet, le nombre infini de solutions ainsi que l'unicité de la solution fondamentale de l'oscillateur apparaissent de façon naturelle dans l'approche algébrique, alors que dans l'approche différentielle, il s'agit essentiellement d'une constatation.

Puisque l'article [3] développe cette approche en détail, nous ne ferons que relever quelques points particulièrement importants de cette méthode.

Premièrement, la solution algébrique permet de contourner quelque peu les problèmes reliés à l'utilisation d'un module bicomplexe de dimension infinie. En effet, dans la méthode différentielle, il semble nécessaire de postuler un module de dimension infinie dans lequel les opérateurs H, X et P agissent pour trouver les fonctions d'onde. Cependant, pour que le problème soit cohérent, il faut que ce module de dimension infinie ait les propriétés d'un espace de Hilbert, en particulier la propriété de complétude. On en vient donc à devoir démontrer la convergence de suites de Cauchy bicomplexes, ce qui n'est pas nécessairement évident. Un autre problème de taille relié à la dimension infinie du module est de démontrer que l'action des opérateurs H, X et P est bien fermée sur le module.

Cependant, en ce qui a trait à la solution algébrique, il semble exister au moins une façon de contourner certains, sinon la majorité, des problèmes liés à la dimension infinie du module, dont les problèmes de convergence. En effet, et c'est ce que nous faisons dans [3], il est possible de résoudre le problème par construction. Dans cette approche, on effectue un certain nombre d'hypothèses [3, sec. 3.1], plus ou moins raisonnables, à propos de notre modèle. On peut alors résoudre le problème sur la base de ces hypothèses. Finalement, une fois les solutions du problème obtenues, on vérifie que les hypothèses de départ sont bien respectées. À noter que ce dernier point, la vérification des hypothèses de départ, est absolument crucial pour que l'approche par construction soit cohérente.

Procédant de cette façon, deux conséquences très importantes ressortent de l'approche par construction, soit le nombre infini de solutions et donc la dimension infinie du module bicomplexe, et la fermeture des opérateurs H, X et P sur le module considéré. En ce sens, l'approche par construction de la solution algébrique permet de trouver les valeurs propres et les fonctions propres de l'oscillateur harmonique bicomplexe, et ce, de façon cohérente et en évitant les questions de convergence et de dimension infinie.

Les calculs préliminaires ainsi que les premières versions de l'article ont été rédigés par RGL sous la supervision judicieuse de LM et DR. Des améliorations de toutes sortes ont été apportées par les trois auteurs (principalement LM et DR) et la version soumise a été peaufinée et essentiellement reformulée par LM.

The Bicomplex Quantum Harmonic Oscillator

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Abstract

The problem of the quantum harmonic oscillator is investigated in the framework of bicomplex numbers. Starting with the commutator of the bicomplex position and momentum operators, we find eigenvalues and eigenvectors of the bicomplex harmonic oscillator Hamiltonian. We construct an infinite-dimensional bicomplex module from the eigenkets of the Hamiltonian. Coordinate-basis eigenfunctions of the bicomplex harmonic oscillator Hamiltonian are obtained in terms of hyperbolic Hermite polynomials.

1 Introduction

The mathematical structure of quantum mechanics consists in Hilbert spaces defined over the field of complex numbers [1]. This structure has been extremely successful in explaining vast amounts of experimental data pertaining largely, but not exclusively, to the world of molecular, atomic and subatomic phenomena.

That success has led a number of investigators, over many decades, to look for general principles that would lead quite inescapably to the complex Hilbert space structure. In recent years, some of these efforts have focused on information-theoretic principles [2, 3]. The fact is, however, that there is no compelling argument restricting the number system on which quantum mechanics is built to the field of complex numbers. A possible extension of quantum mechanics to the field of quaternions was pointed out long ago by Birkhoff and von Neumann [4], and it has since been developed substantially [5, 6].

The fields of real ($\mathbb{R}$) and complex ($\mathbb{C}$) numbers, together with the (noncommutative) field of quaternions ($\mathbb{H}$), share two properties thought to be very important for building a quantum mechanics. Firstly, they are the only associative division algebras over the reals [7]. A *division algebra* is one that has no zero divisors, that is, no nonzero elements w and w' such that $ww' = 0$. Secondly, they are the only associative absolute valued algebras with unit over the reals [8]. An *absolute valued algebra* is one that has a mapping $N(w)$ into $\mathbb{R}$ that satisfies

i. $N(0) = 0$;

ii. $N(w) > 0$ if $w \neq 0$;

iii. $N(aw) = |a|N(w)$ if $a \in \mathbb{R}$;

iv. $N(w_1 + w_2) \leq N(w_1) + N(w_2)$;

v. $N(w_1 w_2) = N(w_1)N(w_2)$.

Property (v), in particular, is widely believed crucial to represent quantum-mechanical probabilities and the correspondence principle with classical mechanics.

Yet several investigations have been carried out on structures sharing some characteristics of quantum mechanics and based on number systems that are neither division nor absolute valued algebras [9, 10]. Of these number systems the ring $\mathbb{T}$ of bicomplex numbers is among the simplest. It has already been shown [11] that structures analogous to bras, kets and Hermitian operators can be defined in finite-dimensional modules over $\mathbb{T}$.

In this paper we intend to pursue that investigation further by extending to bicomplex numbers the problem of the quantum harmonic oscillator. The harmonic oscillator is one of the simplest and, at the same time, one of the most important systems of quantum mechanics, involving as it is an infinite-dimensional vector space.

In section 2 we review the main properties of bicomplex numbers that we will use, together with the notions of module, scalar product and linear operator. Section 3 is devoted to the determination of eigenvalues and eigenkets of the bicomplex quantum harmonic oscillator Hamiltonian, along lines very similar to the algebraic treatment of the usual quantum-mechanical problem. An infinite-dimensional module over $\mathbb{T}$ is explicitly constructed with eigenkets as basis. Section 4 develops the coordinate-basis eigenfunctions associated with the eigenkets obtained. This leads to a straightforward and rather elegant generalization of the usual Hermite polynomials, some of which are plotted explicitly. Section 5 connects with standard quantum mechanics and opens up on new problems.

2 Bicomplex numbers and modules

This section summarizes basic properties of bicomplex numbers and finite-dimensional modules defined over them. The notions of scalar product and linear operators are also introduced. Proofs and additional material can be found in [11, 12, 13, 14].

2.1 Algebraic properties of bicomplex numbers

The set $\mathbb{T}$ of *bicomplex numbers* is defined as

$$\mathbb{T} := \{w = w_e + w_{\mathrm{i}_1}\mathrm{i}_1 + w_{\mathrm{i}_2}\mathrm{i}_2 + w_j\mathrm{j} \mid w_e, w_{\mathrm{i}_1}, w_{\mathrm{i}_2}, w_j \in \mathbb{R}\}, \tag{2.1}$$

where i_1, i_2 and j are imaginary and hyperbolic units such that $\mathrm{i}_1^2 = -1 = \mathrm{i}_2^2$ and $\mathrm{j}^2 = 1$. The product of units is commutative and defined as

$$\mathrm{i}_1\mathrm{i}_2 = \mathrm{j}, \qquad \mathrm{i}_1\mathrm{j} = -\mathrm{i}_2, \qquad \mathrm{i}_2\mathrm{j} = -\mathrm{i}_1. \tag{2.2}$$

With the addition and multiplication of two bicomplex numbers defined in the obvious way, the set $\mathbb{T}$ makes up a commutative ring.

Three important subsets of $\mathbb{T}$ can be specified as

$$\mathbb{C}(\mathrm{i}_k) := \{x + y\mathrm{i}_k \mid x, y \in \mathbb{R}\}, \quad k = 1, 2; \tag{2.3}$$

$$\mathbb{D} := \{x + y\mathrm{j} \mid x, y \in \mathbb{R}\}. \tag{2.4}$$

Each of the sets $\mathbb{C}(\mathrm{i}_k)$ is isomorphic to the field of complex numbers, and $\mathbb{D}$ is the set of *hyperbolic numbers*. An arbitrary bicomplex number w can be written as $w = z + z'\mathrm{i}_2$, where $z = w_e + w_{\mathrm{i}_1}\mathrm{i}_1$ and $z' = w_{\mathrm{i}_2} + w_j\mathrm{i}_1$ both belong to $\mathbb{C}(\mathrm{i}_1)$.

Bicomplex algebra is considerably simplified by the introduction of two bicomplex numbers $\mathbf{e_1}$ and $\mathbf{e_2}$ defined as

$$\mathbf{e_1} := \frac{1+\mathrm{j}}{2}, \qquad \mathbf{e_2} := \frac{1-\mathrm{j}}{2}. \tag{2.5}$$

One easily checks that

$$\mathbf{e_1}^2 = \mathbf{e_1}, \qquad \mathbf{e_2}^2 = \mathbf{e_2}, \qquad \mathbf{e_1} + \mathbf{e_2} = 1, \qquad \mathbf{e_1}\mathbf{e_2} = 0. \tag{2.6}$$

Any bicomplex number w can be written uniquely as

$$w = z_1\mathbf{e_1} + z_2\mathbf{e_2}, \tag{2.7}$$

where z_1 and z_2 both belong to $\mathbb{C}(\mathrm{i}_1)$. Specifically,

$$z_1 = (w_e + w_j) + (w_{\mathrm{i}_1} - w_{\mathrm{i}_2})\mathrm{i}_1, \qquad z_2 = (w_e - w_j) + (w_{\mathrm{i}_1} + w_{\mathrm{i}_2})\mathrm{i}_1. \tag{2.8}$$

The numbers $\mathbf{e_1}$ and $\mathbf{e_2}$ make up the so-called *idempotent basis* of the bicomplex numbers. Note that the last of (2.6) illustrates the fact that $\mathbb{T}$ has zero divisors which, we recall, are nonzero elements whose product is zero.

With w written as in (2.7), we define two projection operators P_1 and P_2 so that

$$P_1(w) = z_1, \qquad P_2(w) = z_2. \tag{2.9}$$

One can easily check that, for $k = 1, 2$,

$$[P_k]^2 = P_k, \qquad \mathbf{e_1}P_1 + \mathbf{e_2}P_2 = \mathrm{Id} \tag{2.10}$$

and that, for any $s, t \in \mathbb{T}$,

$$P_k(s + t) = P_k(s) + P_k(t), \qquad P_k(s \cdot t) = P_k(s) \cdot P_k(t). \tag{2.11}$$

We define the conjugate $w^\dagger$ of the bicomplex number $w = z_1\mathbf{e_1} + z_2\mathbf{e_2}$ as

$$w^\dagger := \bar{z}_1\mathbf{e_1} + \bar{z}_2\mathbf{e_2}, \tag{2.12}$$

where the bar denotes the usual complex conjugation. Operation $w^\dagger$ was denoted by $w^{\dagger_3}$ in [11, 14], consistent with the fact that at least two other types of conjugation can be defined with bicomplex numbers. Making use of (2.6), we immediately see that

$$w \cdot w^\dagger = z_1\bar{z}_1\mathbf{e_1} + z_2\bar{z}_2\mathbf{e_2}. \tag{2.13}$$

Furthermore, for any $s, t \in \mathbb{T}$,

$$(s + t)^\dagger = s^\dagger + t^\dagger, \qquad \left(s^\dagger\right)^\dagger = s, \qquad (s \cdot t)^\dagger = s^\dagger \cdot t^\dagger. \tag{2.14}$$

The real modulus $|w|$ of a bicomplex number w can be defined as

$$|w| := \sqrt{w_e^2 + w_{\mathrm{i}_1}^2 + w_{\mathrm{i}_2}^2 + w_j^2} = \sqrt{(z_1\bar{z}_1 + z_2\bar{z}_2)/2}\,. \tag{2.15}$$

This coincides with the Euclidean norm on $\mathbb{R}^4$. Clearly, $|w| \geq 0$, with $|w| = 0$ if and only if $w = 0$. Moreover, one can show [13] that for any $s, t \in \mathbb{T}$,

$$|s + t| \leq |s| + |t|, \qquad |s \cdot t| \leq \sqrt{2}|s| \cdot |t|. \tag{2.16}$$

The product of two bicomplex numbers w and w' can be written in the idempotent basis as

$$w \cdot w' = (z_1\mathbf{e_1} + z_2\mathbf{e_2}) \cdot (z_1'\mathbf{e_1} + z_2'\mathbf{e_2}) = z_1z_1'\mathbf{e_1} + z_2z_2'\mathbf{e_2}. \tag{2.17}$$

Since 1 is uniquely decomposed as $\mathbf{e_1} + \mathbf{e_2}$, we can see that $w \cdot w' = 1$ if and only if $z_1z_1' = 1 = z_2z_2'$. Thus w has an inverse if and only if $z_1 \neq 0 \neq z_2$, and the inverse w^{-1} is then equal to $z_1^{-1}\mathbf{e_1} + z_2^{-1}\mathbf{e_2}$. A nonzero w that does not have an inverse has the property that either $z_1 = 0$ or $z_2 = 0$, and such a w is a divisor of zero. Zero divisors make up the so-called null cone $\mathcal{NC}$. That terminology comes from the fact that when w is written as $z + z'\mathrm{i}_2$, zero divisors are such that $z^2 + (z')^2 = 0$.

In the idempotent basis, any hyperbolic number can be written as $x_1\mathbf{e_1} + x_2\mathbf{e_2}$, with x_1 and x_2 in $\mathbb{R}$. We define the set $\mathbb{D}^+$ of positive hyperbolic numbers as

$$\mathbb{D}^+ := \{x_1\mathbf{e_1} + x_2\mathbf{e_2} \mid x_1, x_2 \in \mathbb{R}^+\}. \tag{2.18}$$

Clearly, $w \cdot w^\dagger \in \mathbb{D}^+$ for any w in $\mathbb{T}$. We shall say that w is in $\mathbf{e_1}\mathbb{R}^+$ if $w = x_1\mathbf{e_1}$ and x_1 is in $\mathbb{R}^+$ (and similarly with $\mathbf{e_2}\mathbb{R}^+$).

2.2 Modules, scalar product and linear operators

By definition, a vector space is specified over a field of numbers. Bicomplex numbers make up a ring rather than a field, and the structure analogous to a vector space is then a *module*. For later reference we define a $\mathbb{T}$-module M as a set of elements $|\psi\rangle$, $|\phi\rangle$, $|\chi\rangle$, ..., endowed with operations of addition and scalar multiplication, such that the following always holds:

i. $|\psi\rangle + |\phi\rangle = |\phi\rangle + |\psi\rangle$;

ii. $(|\psi\rangle + |\phi\rangle) + |\chi\rangle = |\psi\rangle + (|\phi\rangle + |\chi\rangle)$;

iii. There exists a $|0\rangle$ in M such that $|0\rangle + |\psi\rangle = |\psi\rangle$;

iv. $0 \cdot |\psi\rangle = |0\rangle$;

v. $1 \cdot |\psi\rangle = |\psi\rangle$;

vi. $s \cdot (|\psi\rangle + |\phi\rangle) = s \cdot |\psi\rangle + s \cdot |\phi\rangle$;

vii. $(s + t) \cdot |\psi\rangle = s \cdot |\psi\rangle + t \cdot |\psi\rangle$;

viii. $(st) \cdot |\psi\rangle = s \cdot (t \cdot |\psi\rangle)$.

Here $s, t \in \mathbb{T}$. We have introduced Dirac's notation for elements of M, which we shall call *kets* even though they are not genuine vectors.

A finite-dimensional *free* $\mathbb{T}$-module is a $\mathbb{T}$-module with a finite linearly independent basis. That is, M is a finite-dimensional free $\mathbb{T}$-module if there exist n linearly independent kets $|u_l\rangle$ such that any element $|\psi\rangle$ of M can be written as

$$|\psi\rangle = \sum_{l=1}^{n} w_l |u_l\rangle, \tag{2.19}$$

with $w_l \in \mathbb{T}$. An important subset V of M is the set of all kets for which all w_l in (2.19) belong to $\mathbb{C}(\mathrm{i}_1)$. It was shown in [11] that V is a vector space over the complex numbers, and that any $|\psi\rangle \in M$ can be decomposed uniquely as

$$|\psi\rangle = \mathbf{e_1} P_1(|\psi\rangle) + \mathbf{e_2} P_2(|\psi\rangle), \tag{2.20}$$

where P_1 and P_2 are projectors from M to V. One can show that ket projectors and idempotent-basis projectors (denoted with the same symbol) satisfy the following, for $k = 1, 2$:

$$P_k\left(s|\psi\rangle + t|\phi\rangle\right) = P_k\left(s\right) P_k\left(|\psi\rangle\right) + P_k\left(t\right) P_k\left(|\phi\rangle\right). \tag{2.21}$$

It will be very useful to rewrite (2.7) and (2.20) as

$$w = w_{\mathbf{1}} + w_{\mathbf{2}}, \qquad |\psi\rangle = |\psi\rangle_{\mathbf{1}} + |\psi\rangle_{\mathbf{2}}, \tag{2.22}$$

where

$$w_{\mathbf{1}} = \mathbf{e_1} z_1, \quad w_{\mathbf{2}} = \mathbf{e_2} z_2, \quad |\psi\rangle_{\mathbf{1}} = \mathbf{e_1} P_1(|\psi\rangle), \quad |\psi\rangle_{\mathbf{2}} = \mathbf{e_2} P_2(|\psi\rangle). \tag{2.23}$$

Henceforth bold indices (like **1** and **2**) will always denote objects which include a factor $\mathbf{e_1}$ or $\mathbf{e_2}$, and therefore satisfy an equation like $w_{\mathbf{1}} = \mathbf{e_1} w_{\mathbf{1}}$.

A *bicomplex scalar product* maps two arbitrary kets $|\psi\rangle$ and $|\phi\rangle$ into a bicomplex number $(|\psi\rangle, |\phi\rangle)$, so that the following always holds ($s \in \mathbb{T}$):

i. $(|\psi\rangle, |\phi\rangle + |\chi\rangle) = (|\psi\rangle, |\phi\rangle) + (|\psi\rangle, |\chi\rangle)$;

ii. $(|\psi\rangle, s|\phi\rangle) = s(|\psi\rangle, |\phi\rangle)$;

iii. $(|\psi\rangle, |\phi\rangle) = (|\phi\rangle, |\psi\rangle)^\dagger$;

iv. $(|\psi\rangle, |\psi\rangle) = 0 \;\Leftrightarrow\; |\psi\rangle = 0$.

Property (iii) implies that $(|\psi\rangle, |\psi\rangle) \in \mathbb{D}$, while properties (ii) and (iii) together imply that $(s|\psi\rangle, |\phi\rangle) = s^\dagger(|\psi\rangle, |\phi\rangle)$. One easily shows that

$$(|\psi\rangle, |\phi\rangle) = (|\psi\rangle_{\mathbf{1}}, |\phi\rangle_{\mathbf{1}}) + (|\psi\rangle_{\mathbf{2}}, |\phi\rangle_{\mathbf{2}}). \tag{2.24}$$

Note that

$$(|\psi\rangle_{\mathbf{1}}, |\phi\rangle_{\mathbf{1}})_{\mathbf{1}} = (|\psi\rangle_{\mathbf{1}}, |\phi\rangle_{\mathbf{1}}) \quad \text{and} \quad (|\psi\rangle_{\mathbf{2}}, |\phi\rangle_{\mathbf{2}})_{\mathbf{2}} = (|\psi\rangle_{\mathbf{2}}, |\phi\rangle_{\mathbf{2}}). \tag{2.25}$$

A bicomplex linear operator A is a mapping from M to M such that, for any $s, t \in \mathbb{T}$ and any $|\psi\rangle, |\phi\rangle \in M$

$$A(s|\psi\rangle + t|\phi\rangle) = sA|\psi\rangle + tA|\phi\rangle. \tag{2.26}$$

The bicomplex *adjoint* operator A^* of A is the operator defined so that for any $|\psi\rangle, |\phi\rangle \in M$

$$(|\psi\rangle, A|\phi\rangle) = (A^*|\psi\rangle, |\phi\rangle). \tag{2.27}$$

One can show that in finite-dimensional free $\mathbb{T}$-modules, the adjoint always exists, is linear and satisfies

$$(A^*)^* = A, \quad (sA + tB)^* = s^\dagger A^* + t^\dagger B^*, \quad (AB)^* = B^* A^*. \tag{2.28}$$

A bicomplex linear operator A can always be written as $A = A_{\mathbf{1}} + A_{\mathbf{2}}$, with $A_{\mathbf{1}} = \mathbf{e_1} A$ and $A_{\mathbf{2}} = \mathbf{e_2} A$. Clearly,

$$A|\psi\rangle = A_{\mathbf{1}}|\psi\rangle_{\mathbf{1}} + A_{\mathbf{2}}|\psi\rangle_{\mathbf{2}}. \tag{2.29}$$

We shall say that a ket $|\psi\rangle$ belongs to the null cone if either $|\psi\rangle_{\mathbf{1}} = 0$ or $|\psi\rangle_{\mathbf{2}} = 0$, and that a linear operator A belongs to the null cone if either $A_{\mathbf{1}} = 0$ or $A_{\mathbf{2}} = 0$.

A *self-adjoint* operator is a linear operator H such that $H = H^*$. An operator is self-adjoint if and only if

$$(|\psi\rangle, H|\phi\rangle) = (H|\psi\rangle, |\phi\rangle) \tag{2.30}$$

for all $|\psi\rangle$ and $|\phi\rangle$ in M.

It was shown in [11] that the eigenvalues of a self-adjoint operator acting in a finite-dimensional free $\mathbb{T}$-module, associated with eigenkets not in the null cone, are hyperbolic numbers. One can show quite straightforwardly that two such eigenkets of such a self-adjoint operator, whose eigenvalues differ by a quantity that is not in the null cone, are orthogonal. The proof of this statement will be part of a forthcoming detailed study of finite-dimensional free $\mathbb{T}$-modules [15].

3 The harmonic oscillator

The harmonic oscillator is one of the most widely discussed and widely applied problems in standard quantum mechanics. It is specified as follows: Find the eigenvalues and eigenvectors of a self-adjoint operator H defined as

$$H = \frac{1}{2m} P^2 + \frac{1}{2} m\omega^2 X^2, \tag{3.1}$$

where m and ω are positive real numbers and X and P are self-adjoint operators satisfying the following commutation relation (with i_1 the usual imaginary i):

$$[X, P] = \mathrm{i}_1 \hbar I. \tag{3.2}$$

The problem can be solved exactly either by algebraic [16, 17] or differential [18] methods. In this section we shall show that, viewed as an algebraic problem, the standard quantum-mechanical harmonic oscillator generalizes to bicomplex numbers. In so doing we shall build explicitly an example of an infinite-dimensional free $\mathbb{T}$-module.

3.1 Definitions and assumptions

To state and solve the problem of the bicomplex quantum harmonic oscillator, we start with the following assumptions:

a. Three linear operators X, P and H, related by (3.1), act in a free $\mathbb{T}$-module M.

b. X, P and H are self-adjoint with respect to a scalar product yet to be defined. This means that $(|\psi\rangle, H|\phi\rangle) = (H|\psi\rangle, |\phi\rangle)$ for any $|\psi\rangle$ and $|\phi\rangle$ in M, and similarly with X and P.

c. The scalar product of a ket with itself belongs to $\mathbb{D}^+$.

d. $[X, P] = \mathrm{i}_1\hbar\xi I$, where $\xi \in \mathbb{T}$ is not in the null cone and I is the identity operator on M.

e. There is at least one normalizable eigenket $|E\rangle$ of H which is not in the null cone and whose corresponding eigenvalue E is not in the null cone.

f. Eigenkets of H that are not in the null cone and that correspond to eigenvalues whose difference is not in the null cone are orthogonal.

The consistency of these assumptions will be verified explicitly once the full structure has been obtained. The simplest extension of the canonical commutation relations seems to be embodied in (d). Note that (d) implies that neither X nor P are in the null cone, for if one of them were, ξ would also belong to $\mathcal{NC}$. Assumption (e) implies that H is not in the null cone, and it is necessary to end up with a nontrivial generalization of the standard quantum-mechanical case.

The self-adjointness of X and P implies that the bicomplex number ξ in (d) is in fact hyperbolic. Indeed let $|E\rangle$ be the eigenket of H introduced in (e). By the properties of the scalar product and definition of self-adjointness,

$$\begin{aligned}
\mathrm{i}_1\hbar\xi(|E\rangle, |E\rangle) &= (|E\rangle, \mathrm{i}_1\hbar\xi I|E\rangle) = (|E\rangle, (XP - PX)|E\rangle) \\
&= ((PX - XP)|E\rangle, |E\rangle) = (-\mathrm{i}_1\hbar\xi|E\rangle, |E\rangle) \\
&= \mathrm{i}_1\hbar\xi^\dagger(|E\rangle, |E\rangle).
\end{aligned} \tag{3.3}$$

Since $|E\rangle$ is normalizable, $(|E\rangle, |E\rangle)$ is not in the null cone, and it immediately follows that $\xi = \xi^\dagger$. That is, $\xi = \xi_1\mathbf{e_1} + \xi_2\mathbf{e_2}$, with ξ_1 and ξ_2 real.

Is it possible to further restrict meaningful values of ξ, for instance by a simple rescaling of X and P? To answer this question, let us write

$$X = (\alpha_1\mathbf{e_1} + \alpha_2\mathbf{e_2})X', \qquad P = (\beta_1\mathbf{e_1} + \beta_2\mathbf{e_2})P', \tag{3.4}$$

with nonzero α_k and β_k ($k = 1, 2$). For X' and P' to be self-adjoint, α_k and β_k must be real. Making use of (3.1) we find that

$$\begin{aligned} H &= \frac{1}{2m}(\beta_1^2\mathbf{e_1} + \beta_2^2\mathbf{e_2})(P')^2 + \frac{1}{2}m\omega^2(\alpha_1^2\mathbf{e_1} + \alpha_2^2\mathbf{e_2})(X')^2 \\ &= \frac{1}{2m'}(P')^2 + \frac{1}{2}m'(\omega')^2(X')^2. \end{aligned} \tag{3.5}$$

For m' and ω' to be positive real numbers, $\alpha_1^2\mathbf{e_1} + \alpha_2^2\mathbf{e_2}$ and $\beta_1^2\mathbf{e_1} + \beta_2^2\mathbf{e_2}$ must also belong to $\mathbb{R}^+$. This entails that $\alpha_1^2 = \alpha_2^2$ and $\beta_1^2 = \beta_2^2$, or equivalently $\alpha_1 = \pm\alpha_2$ and $\beta_1 = \pm\beta_2$. Hence we can write

$$\begin{aligned} \mathrm{i}_1\hbar(\xi_1\mathbf{e_1} + \xi_2\mathbf{e_2})I &= [X, P] \\ &= [(\alpha_1\mathbf{e_1} + \alpha_2\mathbf{e_2})X', (\beta_1\mathbf{e_1} + \beta_2\mathbf{e_2})P'] \\ &= (\alpha_1\beta_1\mathbf{e_1} + \alpha_2\beta_2\mathbf{e_2})[X', P']. \end{aligned} \tag{3.6}$$

But this in turn implies that

$$[X', P'] = \mathrm{i}_1\hbar\left(\frac{\xi_1}{\alpha_1\beta_1}\mathbf{e_1} + \frac{\xi_2}{\alpha_2\beta_2}\mathbf{e_2}\right)I = \mathrm{i}_1\hbar(\xi_1'\mathbf{e_1} + \xi_2'\mathbf{e_2})I. \tag{3.7}$$

This equation shows that α_1, α_2, β_1 and β_2 can always be picked so that ξ_1' and ξ_2' are positive. Furthermore, we can choose α_1 and β_1 so as to make ξ_1' equal to 1. But since $|\alpha_1\beta_1| = |\alpha_2\beta_2|$, we have no control over the norm of ξ_2'. The upshot is that we can always write H as in (3.1), with the commutation relation of X and P given by

$$[X, P] = \mathrm{i}_1\hbar\xi I = \mathrm{i}_1\hbar(\xi_1\mathbf{e_1} + \xi_2\mathbf{e_2})I, \qquad \xi_1, \xi_2 \in \mathbb{R}^+. \tag{3.8}$$

We also have the freedom of setting either $\xi_1 = 1$ or $\xi_2 = 1$, but not both.

Just as in the case of the standard quantum harmonic oscillator, we now introduce two operators A and A^* as

$$A := \frac{1}{\sqrt{2m\hbar\omega}}(m\omega X + \mathrm{i}_1 P), \tag{3.9}$$

$$A^* := \frac{1}{\sqrt{2m\hbar\omega}}(m\omega X - \mathrm{i}_1 P). \tag{3.10}$$

Since P is self-adjoint, one always has $(-\mathrm{i}_1 P|\psi\rangle, |\phi\rangle) = (|\psi\rangle, \mathrm{i}_1 P|\phi\rangle)$, which means that the adjoint of $\mathrm{i}_1 P$ is $-\mathrm{i}_1 P$. This implies that, as the notation suggests, A^* is indeed the adjoint of A. Equations (3.9) and (3.10) can be inverted as

$$X = \sqrt{\frac{\hbar}{2m\omega}}(A + A^*), \qquad P = -\mathrm{i}_1\sqrt{\frac{\hbar m\omega}{2}}(A - A^*). \tag{3.11}$$

The commutator of A and A^* is given by

$$[A, A^*] = \frac{1}{2m\hbar\omega}\left\{[\mathrm{i}_1 P, m\omega X] + [m\omega X, -\mathrm{i}_1 P]\right\} = \xi I. \tag{3.12}$$

Substituting (3.11) in (3.1), one easily finds that

$$H = \hbar\omega\left(A^*A + \frac{\xi}{2}I\right) = \hbar\omega\left(AA^* - \frac{\xi}{2}I\right). \tag{3.13}$$

From (3.12) and (3.13), the following commutation relations are straightforwardly obtained:

$$[H, A] = -\hbar\omega\xi A, \qquad [H, A^*] = \hbar\omega\xi A^*. \tag{3.14}$$

3.2 Eigenkets and eigenvalues of H

From assumption (e) we know that there is a normalizable ket $|E\rangle$ such that

$$H|E\rangle = E|E\rangle. \tag{3.15}$$

We can write

$$H = H_{\mathbf{1}} + H_{\mathbf{2}}, \tag{3.16}$$

$$E = E_{\mathbf{1}} + E_{\mathbf{2}}, \tag{3.17}$$

$$|E\rangle = |E\rangle_{\mathbf{1}} + |E\rangle_{\mathbf{2}}, \tag{3.18}$$

where $E_{\mathbf{1}} = \mathbf{e_1}E$, etc. Assumption (e) implies that none of the quantities in (3.16)–(3.18) vanishes. Substitution of these equations in (3.15) immediately yields

$$H_{\mathbf{1}}|E\rangle_{\mathbf{1}} = E_{\mathbf{1}}|E\rangle_{\mathbf{1}}, \qquad H_{\mathbf{2}}|E\rangle_{\mathbf{2}} = E_{\mathbf{2}}|E\rangle_{\mathbf{2}}. \tag{3.19}$$

Following the treatment made in standard quantum mechanics, we now apply operators HA and HA^* on $|E\rangle$. Making use of (3.14) we get

$$HA|E\rangle = (AH + [H, A])|E\rangle = (E - \hbar\omega\xi)A|E\rangle, \tag{3.20}$$

$$HA^*|E\rangle = (A^*H + [H, A^*])|E\rangle = (E + \hbar\omega\xi)A^*|E\rangle. \tag{3.21}$$

We see that if $A|E\rangle$ does not vanish, it is an eigenket of H with eigenvalue $E - \hbar\omega\xi$. Similarly, unless $A^*|E\rangle$ vanishes, it is an eigenket of H with eigenvalue $E + \hbar\omega\xi$.

Let l be a positive integer. We will show by induction that unless $A^l|E\rangle$ vanishes, it is an eigenket of H with eigenvalue $E - l\hbar\omega\xi$. We have just shown that this is true for $l = 1$. Let it be true for $l - 1$. We have

$$\begin{aligned} HA^l|E\rangle &= HAA^{l-1}|E\rangle = (AHA^{l-1} + [H, A]A^{l-1})|E\rangle \\ &= A(E - (l-1)\hbar\omega\xi)A^{l-1}|E\rangle - \hbar\omega\xi AA^{l-1}|E\rangle \\ &= \{E - l\hbar\omega\xi\}\, A^l|E\rangle, \end{aligned} \tag{3.22}$$

which proves the claim. Similarly, unless $(A^*)^l|E\rangle$ vanishes, it is an eigenket of H with eigenvalue $E + l\hbar\omega\xi$, that is,

$$H(A^*)^l|E\rangle = (E + l\hbar\omega\xi)(A^*)^l|E\rangle. \tag{3.23}$$

Equations (3.22) and (3.23) separate in the idempotent basis. Multiplying them by $\mathbf{e_k}$ and using the fact that $HA^l = H_\mathbf{1}A_\mathbf{1}^l + H_\mathbf{2}A_\mathbf{2}^l$, we easily find that $(k = 1, 2)$

$$H_\mathbf{k}A_\mathbf{k}^l|E\rangle_\mathbf{k} = (E_\mathbf{k} - l\hbar\omega\xi_\mathbf{k})A_\mathbf{k}^l|E\rangle_\mathbf{k}, \tag{3.24}$$
$$H_\mathbf{k}(A_\mathbf{k}^*)^l|E\rangle_\mathbf{k} = (E_\mathbf{k} + l\hbar\omega\xi_\mathbf{k})(A_\mathbf{k}^*)^l|E\rangle_\mathbf{k}. \tag{3.25}$$

Consistent with the bold notation, we have written $\xi_\mathbf{1} = \mathbf{e_1}\xi_1$ and $\xi_\mathbf{2} = \mathbf{e_2}\xi_2$.

We now prove the following lemma.

Lemma 1 *Let $|\phi\rangle$ be an eigenket of H associated with the (finite) eigenvalue λ. Then,*

$$(A|\phi\rangle, A|\phi\rangle) = \left\{\frac{\lambda}{\hbar\omega} - \frac{\xi}{2}\right\}(|\phi\rangle, |\phi\rangle) \tag{3.26}$$

and

$$(A^*|\phi\rangle, A^*|\phi\rangle) = \left\{\frac{\lambda}{\hbar\omega} + \frac{\xi}{2}\right\}(|\phi\rangle, |\phi\rangle). \tag{3.27}$$

Proof.

Making use of (3.13) we have

$$\begin{aligned}(A|\phi\rangle, A|\phi\rangle) = (|\phi\rangle, A^*A|\phi\rangle) &= \left(|\phi\rangle, \left\{\frac{H}{\hbar\omega} - \frac{\xi}{2}I\right\}|\phi\rangle\right)\\ &= \left(|\phi\rangle, \left\{\frac{\lambda}{\hbar\omega} - \frac{\xi}{2}\right\}|\phi\rangle\right) = \left\{\frac{\lambda}{\hbar\omega} - \frac{\xi}{2}\right\}(|\phi\rangle, |\phi\rangle).\end{aligned}$$

The proof of the second equality is similar.

□

Two important consequences of lemma 1 are the following. Firstly, whenever $(|\phi\rangle, |\phi\rangle)$ is finite, so are $(A|\phi\rangle, A|\phi\rangle)$ and $(A^*|\phi\rangle, A^*|\phi\rangle)$. And secondly, the lemma also holds when all quantities are replaced by corresponding idempotent projections. That is, for $k = 1, 2$,

$$(A_\mathbf{k}|\phi\rangle_\mathbf{k}, A_\mathbf{k}|\phi\rangle_\mathbf{k}) = \left\{\frac{\lambda_\mathbf{k}}{\hbar\omega} - \frac{\xi_\mathbf{k}}{2}\right\}(|\phi\rangle_\mathbf{k}, |\phi\rangle_\mathbf{k}). \tag{3.28}$$

Let us now apply lemma 1 to the case where $|\phi\rangle_\mathbf{k} = |E\rangle_\mathbf{k}$. Since $(|E\rangle, |E\rangle)$ is in $\mathbb{D}^+$, $(|E\rangle_\mathbf{k}, |E\rangle_\mathbf{k})$ is in $\mathbf{e_k}\mathbb{R}^+$ (and is nonzero). But then (3.28) implies that

$(A_{\mathbf{k}}|E\rangle_{\mathbf{k}}, A_{\mathbf{k}}|E\rangle_{\mathbf{k}})$ is in $\mathbf{e}_{\mathbf{k}}\mathbb{R}^+$ only if $E_{\mathbf{k}}/\hbar\omega - \xi_{\mathbf{k}}/2$ is in $\mathbf{e}_{\mathbf{k}}\mathbb{R}^+$. Let us write (3.28) for the case where $|\phi\rangle_{\mathbf{k}} = A^l_{\mathbf{k}}|E\rangle_{\mathbf{k}}$. Making use of (3.24), we find that

$$\left(A^{l+1}_{\mathbf{k}}|E\rangle_{\mathbf{k}}, A^{l+1}_{\mathbf{k}}|E\rangle_{\mathbf{k}}\right) = \left\{\frac{E_{\mathbf{k}}}{\hbar\omega} - \left(l + \frac{1}{2}\right)\xi_{\mathbf{k}}\right\}\left(A^l_{\mathbf{k}}|E\rangle_{\mathbf{k}}, A^l_{\mathbf{k}}|E\rangle_{\mathbf{k}}\right). \tag{3.29}$$

Again, and assuming that $A^l_{\mathbf{k}}|E\rangle_{\mathbf{k}}$ doesn't vanish, $(A^{l+1}_{\mathbf{k}}|E\rangle_{\mathbf{k}}, A^{l+1}_{\mathbf{k}}|E\rangle_{\mathbf{k}})$ is in $\mathbf{e}_{\mathbf{k}}\mathbb{R}^+$ only if $E_{\mathbf{k}}/\hbar\omega - (l + 1/2)\xi_{\mathbf{k}}$ is in $\mathbf{e}_{\mathbf{k}}\mathbb{R}^+$.

Clearly, however, this cannot go on forever. Let l_k be the smallest positive integer for which

$$P_k\left(\frac{E_{\mathbf{k}}}{\hbar\omega} - \left(l_k + \frac{1}{2}\right)\xi_{\mathbf{k}}\right) \leq 0. \tag{3.30}$$

If the equality holds in (3.30), then (3.29) implies that $A^{l_k+1}_{\mathbf{k}}|E\rangle_{\mathbf{k}} = 0$. If the inequality holds, the same conclusion follows since otherwise the scalar product of a nonzero ket with itself would be outside $\mathbb{D}^+$. The upshot is that

$$A_{\mathbf{k}}|\phi_0\rangle_{\mathbf{k}} = 0 \quad \text{with} \quad |\phi_0\rangle_{\mathbf{k}} = A^{l_k}_{\mathbf{k}}|E\rangle_{\mathbf{k}}. \tag{3.31}$$

Applying $H_{\mathbf{k}}$ obtained from the first part of (3.13) on $|\phi_0\rangle_{\mathbf{k}}$, we get

$$H_{\mathbf{k}}|\phi_0\rangle_{\mathbf{k}} = \hbar\omega\left(A^*_{\mathbf{k}}A_{\mathbf{k}} + \frac{1}{2}\xi_{\mathbf{k}}I\right)|\phi_0\rangle_{\mathbf{k}} = \frac{1}{2}\hbar\omega\xi_{\mathbf{k}}|\phi_0\rangle_{\mathbf{k}}. \tag{3.32}$$

That is, $|\phi_0\rangle_{\mathbf{k}}$ is an eigenket of $H_{\mathbf{k}}$ with eigenvalue $\hbar\omega\xi_{\mathbf{k}}/2$.

Making use of an argument similar to the one leading to (3.25), we can see that the ket $(A^*_{\mathbf{k}})^l|\phi_0\rangle_{\mathbf{k}}$ is an eigenket of $H_{\mathbf{k}}$ with eigenvalue $(l + 1/2)\hbar\omega\xi_{\mathbf{k}}$. For later convenience we define

$$|\phi_l\rangle_{\mathbf{1}} = (l!\xi^l_1)^{-1/2}(A^*_{\mathbf{1}})^l|\phi_0\rangle_{\mathbf{1}}, \qquad |\phi_l\rangle_{\mathbf{2}} = (l!\xi^l_2)^{-1/2}(A^*_{\mathbf{2}})^l|\phi_0\rangle_{\mathbf{2}}. \tag{3.33}$$

Note that ξ_1 and ξ_2, being within an inversion operator, cannot carry bold indices. By the idempotent projection of the second part of lemma 1, $|\phi_l\rangle_{\mathbf{k}}$ does not vanish for any l. We have therefore constructed two infinite sequences of kets, each of which is a sequence of eigenkets of an idempotent projection of H.

We now define

$$|\phi_l\rangle = |\phi_l\rangle_{\mathbf{1}} + |\phi_l\rangle_{\mathbf{2}}. \tag{3.34}$$

It is easy to check that $|\phi_l\rangle$ is an eigenket of H with eigenvalue $(l + 1/2)\hbar\omega\xi$. By assumption (f), $|\phi_l\rangle$ and $|\phi_{l'}\rangle$ are orthogonal if $l \neq l'$.

3.3 Infinite-dimensional free $\mathbb{T}$-module

Let M be the collection of all finite linear combinations of kets $|\phi_l\rangle$, with bicomplex coefficients. That is,

$$M := \left\{ \sum_l w_l |\phi_l\rangle \mid w_l \in \mathbb{T} \right\}. \tag{3.35}$$

It is understood that adding terms with zero coefficients doesn't yield a new ket. Let us define the addition of two elements of M and the multiplication of an element of M by a bicomplex number in the obvious way. Furthermore let us write $|0\rangle = 0 \cdot |\phi_0\rangle$. It is then easy to check that the eight defining properties of a $\mathbb{T}$-module stated in section 2.2 are satisfied. M is therefore a $\mathbb{T}$-module.

If the coefficients w_l in (3.35) are restricted to elements of $\mathbb{C}(\mathrm{i}_1)$, the resulting set V is a vector space over $\mathbb{C}(\mathrm{i}_1)$. It is the analog of the vector space introduced before (2.20), which was used in [11] to define the projection P_k and prove a number of results on finite-dimensional modules.

The scalar product of elements of M has hitherto been specified only partially, in particular by requiring that $|\phi_l\rangle$ and $|\phi_{l'}\rangle$ be orthogonal if $l \neq l'$. We now set

$$(|\phi_0\rangle, |\phi_0\rangle) = 1. \tag{3.36}$$

Equation (3.33) implies that

$$\begin{aligned} |\phi_{l+1}\rangle &= |\phi_{l+1}\rangle_{\mathbf{1}} + |\phi_{l+1}\rangle_{\mathbf{2}} = \mathbf{e_1}|\phi_{l+1}\rangle_{\mathbf{1}} + \mathbf{e_2}|\phi_{l+1}\rangle_{\mathbf{2}} \\ &= \frac{\mathbf{e_1}}{\sqrt{(l+1)\xi_1}} A^*_{\mathbf{1}}|\phi_l\rangle_{\mathbf{1}} + \frac{\mathbf{e_2}}{\sqrt{(l+1)\xi_2}} A^*_{\mathbf{2}}|\phi_l\rangle_{\mathbf{2}} \\ &= \frac{1}{\sqrt{(l+1)\xi}} A^*|\phi_l\rangle. \end{aligned} \tag{3.37}$$

Letting A act on both sides of (3.37) and making use of (3.13), we find that

$$\begin{aligned} A|\phi_{l+1}\rangle &= \frac{1}{\sqrt{(l+1)\xi}} AA^*|\phi_l\rangle = \frac{1}{\sqrt{(l+1)\xi}} \left\{ \frac{H}{\hbar\omega} + \frac{\xi}{2} I \right\} |\phi_l\rangle \\ &= \sqrt{(l+1)\xi}\, |\phi_l\rangle. \end{aligned} \tag{3.38}$$

From (3.37) and the second part of lemma 1 we get

$$(|\phi_{l+1}\rangle, |\phi_{l+1}\rangle) = \frac{1}{(l+1)\xi} (A^*|\phi_l\rangle, A^*|\phi_l\rangle) = (|\phi_l\rangle, |\phi_l\rangle). \tag{3.39}$$

Owing to (3.36), the solution of this recurrence equation is

$$(|\phi_l\rangle, |\phi_l\rangle) = 1, \qquad l = 0, 1, 2, \ldots \tag{3.40}$$

We now fully specify the scalar product of two arbitrary elements $|\psi\rangle$ and $|\chi\rangle$ of M as follows. Let

$$|\psi\rangle = \sum_l w_l|\phi_l\rangle, \qquad |\chi\rangle = \sum_l v_l|\phi_l\rangle. \tag{3.41}$$

The two sums are finite. Without loss of generality, we can let them run over the same set of indices. Indeed this simply amounts to possibly adding terms with zero coefficients in either or both sums. With this we define the scalar product as

$$(|\psi\rangle, |\chi\rangle) := \sum_l w_l^\dagger v_l(|\phi_l\rangle, |\phi_l\rangle) = \sum_l w_l^\dagger v_l. \tag{3.42}$$

With this specification, it is easy to check that the four defining properties of a scalar product stated in section 2.2 are satisfied. Note that the right-hand side of (3.42) is always finite.

Clearly, the kets $|\phi_l\rangle$ generate M. To show that they are linearly independent, we assume that $|\psi\rangle$ defined in (3.41) vanishes. Letting m be one of the l indices, we have

$$0 = (|\phi_m\rangle, |\psi\rangle) = \sum_l w_l\delta_{ml} = w_m. \tag{3.43}$$

Hence $w_m = 0$ for all m, and the linear independence follows. This shows that M is an infinite-dimensional free $\mathbb{T}$-module.

There remains to check the six assumptions made at the beginning of section 3.1. Assumption (a) is obvious, the action of X and P on M being most easily obtained through the action of A and A^*. Similarly with (b), the self-adjointness of X and P follows from the easily verifiable fact that A^* is the adjoint of A on the whole of M. Assumption (c) is an immediate consequence of definition (3.42). Assumption (d) follows from the commutation relation $[A, A^*] = \xi I$. This one is easily checked when acting on eigenkets of H and therefore, by linearity, it holds on any ket. Assumption (e) is satisfied by any ket $|\phi_l\rangle$. There only remains to check assumption (f), which is a little more tricky.

Let $|\psi\rangle$ defined in (3.41) be an eigenket of H with eigenvalue λ. This means that

$$H\sum_l w_l|\phi_l\rangle = \lambda\sum_l w_l|\phi_l\rangle \tag{3.44}$$

which, owing to the linear independence of the $|\phi_l\rangle$, reduces to

$$\left(l + \frac{1}{2}\right)\hbar\omega\xi w_l = \lambda w_l. \tag{3.45}$$

In the idempotent basis this becomes ($k = 1, 2$)

$$\left(l + \frac{1}{2}\right)\hbar\omega\xi_{\mathbf{k}} w_{l\mathbf{k}} = \lambda_{\mathbf{k}} w_{l\mathbf{k}}. \tag{3.46}$$

Let $\lambda_{\mathbf{1}} \neq 0$. Since $\xi_{\mathbf{1}}$ does not vanish, at most one coefficient $w_{l\mathbf{1}}$ does not vanish, for otherwise $\lambda_{\mathbf{1}}$ would satisfy two incompatible equations. If $\lambda_{\mathbf{1}} = 0$, all $w_{l\mathbf{1}}$ vanish. A similar argument holds for **2**. Hence the eigenket of H has the form

$$|\phi\rangle = w_{l\mathbf{1}}|\phi_l\rangle_{\mathbf{1}} + w_{l'\mathbf{2}}|\phi_{l'}\rangle_{\mathbf{2}}, \tag{3.47}$$

with one of the coefficients vanishing if the corresponding $\lambda_{\mathbf{k}}$ vanishes. If both $\lambda_{\mathbf{k}}$ vanish, all $w_{l\mathbf{k}} = 0$ and there is no eigenket. The upshot is that (3.47) represents the most general eigenket of H. Its associated eigenvalue λ is

$$\lambda = \hbar\omega \left\{ \left(l + \frac{1}{2} \right) \xi_1 \mathbf{e_1} + \left(l' + \frac{1}{2} \right) \xi_2 \mathbf{e_2} \right\}. \tag{3.48}$$

It is now a simple matter to check that assumption (f) is satisfied. Note that the restriction on the difference of eigenvalues cannot be dispensed with. Indeed the two kets

$$|\phi\rangle = |\phi_1\rangle_{\mathbf{1}} + |\phi_2\rangle_{\mathbf{2}}, \qquad |\phi'\rangle = |\phi_1\rangle_{\mathbf{1}} + |\phi_3\rangle_{\mathbf{2}} \tag{3.49}$$

are examples of eigenkets that correspond to different eigenvalues whose difference is in the null cone. Clearly, they are not orthogonal.

4 Harmonic oscillator wave functions

4.1 Bicomplex function space

Consider the set S of all square-integrable complex C^∞ functions of a real variable x. Let a bicomplex function $u(x)$ be defined as

$$u(x) = \mathbf{e_1} u_1(x) + \mathbf{e_2} u_2(x). \tag{4.1}$$

We say that u is square integrable and C^∞ if u_1 and u_2 are both in S. It is easy to check that the set of all square-integrable bicomplex C^∞ functions of a real variable is a $\mathbb{T}$-module, which we shall denote by M^∞.

Let $u(x)$ and $v(x)$ both belong to M^∞. We define a mapping (u, v) of this pair of functions into $\mathbb{D}^+$ as follows:

$$(u, v) := \int_{-\infty}^{\infty} u^\dagger(x) v(x) \mathrm{d}x = \int_{-\infty}^{\infty} \left[\mathbf{e_1} \bar{u}_1(x) v_1(x) + \mathbf{e_2} \bar{u}_2(x) v_2(x) \right] \mathrm{d}x. \tag{4.2}$$

It is not hard to see that (4.2) satisfies all the properties of a bicomplex scalar product.

Let $\xi \in \mathbb{D}^+$. We define two operators X and P that act on elements of M^∞ as follows:

$$X\{u(x)\} := xu(x), \qquad P\{u(x)\} := -\mathrm{i}_1 \hbar \xi \frac{\mathrm{d}u(x)}{\mathrm{d}x}. \tag{4.3}$$

It is not difficult to show that $[X, P] = \mathrm{i}_1\hbar\xi I$. Note that the action of X and P on an element of M^∞ doesn't always yield an element of M^∞. This could be fixed by restricting M^∞ further, but we won't need to do this here.

One can easily check that $(Xu, v) = (u, Xv)$, so that X is self-adjoint. The self-adjointness of P can be proved as

$$\begin{aligned}
&(Pu, v) - (u, Pv) \\
&= \int_{-\infty}^{\infty} \left(-\mathrm{i}_1\hbar\xi\frac{\mathrm{d}u(x)}{\mathrm{d}x}\right)^\dagger v(x)\mathrm{d}x - \int_{-\infty}^{\infty} u^\dagger(x)\left(-\mathrm{i}_1\hbar\xi\frac{\mathrm{d}v(x)}{\mathrm{d}x}\right)\mathrm{d}x \\
&= \mathrm{i}_1\hbar\xi\left\{\int_{-\infty}^{\infty}\frac{\mathrm{d}\left[u^\dagger(x)v(x)\right]}{\mathrm{d}x}\mathrm{d}x\right\} \\
&= \mathrm{i}_1\hbar\xi\left[u^\dagger(x)v(x)\right]_{-\infty}^{\infty} = 0.
\end{aligned}$$

The final equality comes from the fact that u and v, being square integrable, vanish at infinity.

4.2 Eigenfunctions of H

Let H be defined as in (3.1), with X and P specified as in (4.3). The eigenvalue equation for H is then given by

$$Hu(x) = -\frac{\hbar^2\xi^2}{2m}\frac{\mathrm{d}^2u(x)}{\mathrm{d}x^2} + \frac{1}{2}m\omega^2x^2u(x) = Eu(x). \tag{4.4}$$

In the idempotent basis this separates into the following two equations ($k = 1, 2$):

$$-\frac{\hbar^2\xi_k^2}{2m}\frac{\mathrm{d}^2u_k(x)}{\mathrm{d}x^2} + \frac{1}{2}m\omega^2x^2u_k(x) = E_ku_k(x). \tag{4.5}$$

Each of these equations is essentially the eigenvalue equation for the Hamiltonian of the standard quantum harmonic oscillator. The only difference is that $\hbar$ is replaced by $\hbar\xi_k$.

The eigenfunction associated with the lowest eigenvalue of (4.5) is given by

$$\phi_{0k}(x) = \left(\frac{m\omega}{\pi\hbar\xi_k}\right)^{1/4}\exp\left\{-\frac{m\omega}{2\hbar\xi_k}x^2\right\}. \tag{4.6}$$

The corresponding eigenfunction of H is therefore given by

$$\begin{aligned}
\phi_0(x) &= \mathbf{e_1}\phi_{01}(x) + \mathbf{e_2}\phi_{02}(x) \\
&= \mathbf{e_1}\left(\frac{m\omega}{\pi\hbar\xi_1}\right)^{1/4}\exp\left\{-\frac{m\omega}{2\hbar\xi_1}x^2\right\} + \mathbf{e_2}\left(\frac{m\omega}{\pi\hbar\xi_2}\right)^{1/4}\exp\left\{-\frac{m\omega}{2\hbar\xi_2}x^2\right\} \\
&= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(\frac{\mathbf{e_1}}{\xi_1^{1/4}} + \frac{\mathbf{e_2}}{\xi_2^{1/4}}\right)\left\{\mathbf{e_1}\exp\left[-\frac{m\omega}{2\hbar\xi_1}x^2\right] + \mathbf{e_2}\exp\left[-\frac{m\omega}{2\hbar\xi_2}x^2\right]\right\}
\end{aligned} \tag{4.7}$$

It can be shown [12] that for any bicomplex number $w = z_1\mathbf{e_1} + z_2\mathbf{e_2}$,

$$\exp\{w\} = \mathbf{e_1}\exp\{z_1\} + \mathbf{e_2}\exp\{z_2\}. \tag{4.8}$$

This holds also for any polynomial function $Q(x)$, that is,

$$Q(z_1\mathbf{e_1} + z_2\mathbf{e_2}) = \mathbf{e_1}Q(z_1) + \mathbf{e_2}Q(z_2). \tag{4.9}$$

Moreover, if $\xi = \xi_1\mathbf{e_1} + \xi_2\mathbf{e_2}$ with ξ_1 and ξ_2 positive, we have

$$\frac{1}{\xi^{1/4}} = \frac{\mathbf{e_1}}{\xi_1^{1/4}} + \frac{\mathbf{e_2}}{\xi_2^{1/4}}. \tag{4.10}$$

Substituting (4.8) and (4.10) in (4.7), we get

$$\phi_0(x) = \left(\frac{m\omega}{\pi\hbar\xi}\right)^{1/4} \exp\left\{-\frac{m\omega}{2\hbar\xi}x^2\right\}. \tag{4.11}$$

From the normalization of ϕ_{01} and ϕ_{02}, we find that

$$\begin{aligned}(\phi_0, \phi_0) &= \int_{-\infty}^{\infty} \left[\mathbf{e_1}\bar{\phi}_{01}(x)\phi_{01}(x) + \mathbf{e_2}\bar{\phi}_{02}(x)\phi_{02}(x)\right] \mathrm{d}x \\ &= \mathbf{e_1} + \mathbf{e_2} = 1. \end{aligned} \tag{4.12}$$

The eigenfunction associated with the lth eigenvalue of (4.5) is given by [19]

$$\phi_{lk}(x) = \left[\sqrt{\frac{m\omega}{\pi\hbar\xi_k}}\frac{1}{2^l l!}\right]^{1/2} \mathrm{e}^{-\theta_k^2/2} H_l(\theta_k), \tag{4.13}$$

where

$$\theta_k = \sqrt{\frac{m\omega}{\hbar\xi_k}}x \tag{4.14}$$

and $H_l(\theta_k)$ is the Hermite polynomial of order l. Just as in (3.34) we now define

$$\phi_l(x) = \mathbf{e_1}\phi_{l1}(x) + \mathbf{e_2}\phi_{l2}(x). \tag{4.15}$$

We therefore obtain

$$\begin{aligned}\phi_l(x) &= \mathbf{e_1}\left[\sqrt{\frac{m\omega}{\pi\hbar\xi_1}}\frac{1}{2^l l!}\right]^{1/2} \mathrm{e}^{-\theta_1^2/2} H_l(\theta_1) + \mathbf{e_2}\left[\sqrt{\frac{m\omega}{\pi\hbar\xi_2}}\frac{1}{2^l l!}\right]^{1/2} \mathrm{e}^{-\theta_2^2/2} H_l(\theta_2) \\ &= \left\{\mathbf{e_1}\left[\sqrt{\frac{m\omega}{\pi\hbar\xi_1}}\frac{1}{2^l l!}\right]^{1/2} + \mathbf{e_2}\left[\sqrt{\frac{m\omega}{\pi\hbar\xi_2}}\frac{1}{2^l l!}\right]^{1/2}\right\} \\ &\qquad \cdot \left\{\mathbf{e_1}\mathrm{e}^{-\theta_1^2/2} + \mathbf{e_2}\mathrm{e}^{-\theta_2^2/2}\right\}\left\{\mathbf{e_1}H_l(\theta_1) + \mathbf{e_2}H_l(\theta_2)\right\}. \end{aligned} \tag{4.16}$$

Letting $\theta := \mathbf{e_1}\theta_1 + \mathbf{e_2}\theta_2$ and making use of (4.8)–(4.10), we finally obtain

$$\phi_l(x) = \left[\sqrt{\frac{m\omega}{\pi\hbar\xi}}\frac{1}{2^l l!}\right]^{1/2} \mathrm{e}^{-\theta^2/2} H_l(\theta), \tag{4.17}$$

where

$$H_l(\theta) := \mathbf{e_1}H_l(\theta_1) + \mathbf{e_2}H_l(\theta_2) \tag{4.18}$$

is a hyperbolic Hermite polynomial of order l.

Equation (4.17) is one of the central results of this paper. It expresses normalized eigenfunctions of the bicomplex harmonic oscillator Hamiltonian purely in terms of hyperbolic constants and functions, with no reference to a particular representation like $\{\mathbf{e_k}\}$. Indeed ξ can be viewed as a $\mathbb{D}^+$ constant, θ is equal to $\sqrt{m\omega/\hbar\xi}\, x$ and $H_l(\theta)$ is just the Hermite polynomial in θ.

Let $\tilde{M}$ be the collection of all finite linear combinations of bicomplex functions $\phi_l(x)$, with bicomplex coefficients. That is,

$$\tilde{M} := \left\{ \sum_l w_l \phi_l(x) \mid w_l \in \mathbb{T} \right\}. \tag{4.19}$$

It is easy to see that $\tilde{M}$ is a submodule of the module M^∞ defined earlier in terms of C^∞ functions, and that $\tilde{M}$ is isomorphic to the module M defined in section 3.3.

In section 3.3, the most general eigenket of H was written as in (3.47). The corresponding eigenfunction has the form

$$\phi(x) = \mathbf{e_1}w_{l1}\phi_{l1}(x) + \mathbf{e_2}w_{l'2}\phi_{l'2}, \tag{4.20}$$

with w_{l1} and $w_{l'2}$ in $\mathbb{C}(\mathrm{i}_1)$. The eigenfunction can be written explicitly as

$$\phi(x) = \left[\frac{m\omega}{\pi\hbar}\right]^{1/4} \left\{ \mathbf{e_1}\frac{w_{l1}\mathrm{e}^{-\theta_1^2/2}}{\sqrt{2^l l! \sqrt{\xi_1}}} H_l(\theta_1) + \mathbf{e_2}\frac{w_{l'2}\mathrm{e}^{-\theta_2^2/2}}{\sqrt{2^{l'}(l')! \sqrt{\xi_2}}} H_{l'}(\theta_2) \right\}. \tag{4.21}$$

The function ϕ is normalized, i.e. $(\phi, \phi) = 1$, if

$$|w_{l1}|^2\mathbf{e_1} + |w_{l'2}|^2\mathbf{e_2} = 1. \tag{4.22}$$

This means that $|w_{l1}| = 1 = |w_{l'2}|$.

The function $\phi(x)$ can also be expressed in terms of the hyperbolic units 1 and j instead of $\mathbf{e_1}$ and $\mathbf{e_2}$. Letting $w_{l1} = 1 = w_{l'2}$, we get

$$\begin{aligned}\phi(x) = \left[\frac{m\omega}{\pi\hbar}\right]^{1/4} \frac{1}{2} \Bigg\{ &\left[\frac{\mathrm{e}^{-\theta_1^2/2}}{\sqrt{2^l l! \sqrt{\xi_1}}} H_l(\theta_1) + \frac{\mathrm{e}^{-\theta_2^2/2}}{\sqrt{2^{l'}(l')! \sqrt{\xi_2}}} H_{l'}(\theta_2) \right] \\ &+ \mathrm{j} \left[\frac{\mathrm{e}^{-\theta_1^2/2}}{\sqrt{2^l l! \sqrt{\xi_1}}} H_l(\theta_1) - \frac{\mathrm{e}^{-\theta_2^2/2}}{\sqrt{2^{l'}(l')! \sqrt{\xi_2}}} H_{l'}(\theta_2) \right] \Bigg\}. \end{aligned} \tag{4.23}$$

5 Discussion

It is instructive to plot some of the functions given in (4.23). At this stage we do not suggest any specific physical interpretation of the bicomplex eigenfunctions. However, it is useful to see how the standard quantum harmonic oscillator is embedded in the bicomplex harmonic oscillator. In all plots we let $\xi_1 = 1$ and take as independent variable $y = \sqrt{m\omega/\hbar}\, x$. Dashed lines represent the real part of ϕ while dotted lines represent the hyperbolic part. Solid lines represent the function $|\phi|^2$, where $|\cdot|$ is the norm defined in (2.15). The normalization factor $(m\omega/\hbar)^{1/4}$ is omitted.

In figure 1 we let $\xi_2 = 1$ and $l = 1 = l'$. The hyperbolic part of ϕ vanishes and the real part is equal to the second lowest eigenfunction of the standard harmonic oscillator.

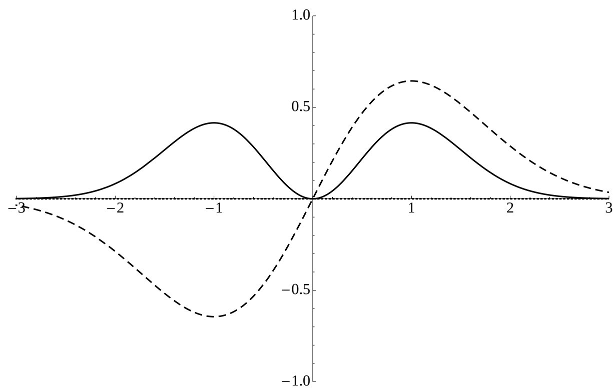

Figure 1: Eigenfunction (4.23) for $\xi_1 = 1 = \xi_2$ and $l = 1 = l'$

In all cases where $\xi_1 = 1 = \xi_2$ and $l = 1 = l'$, we recover the usual harmonic oscillator eigenfunctions. But these can also be recovered in a different way. One can write $w_{l1} = 1$ and $w_{l'2} = 0$ in (4.20), in which case the factor of $\mathbf{e_1}$ coincides with the standard eigenfunction.

In figure 2 we let $\xi_2 = 1$, $l = 1$ and $l' = 2$. There is a nonvanishing hyperbolic part in spite of the fact that $\xi = \mathbf{e_1}\xi_1 + \mathbf{e_2}\xi_2 = 1$, that is, even if X and P have the usual quantum-mechanical commutation relations.

Figure 3 displays a case where $\xi_2 \neq \xi_1$, and therefore where the canonical commutation relations are irreducibly bicomplex. Specifically, $\xi_2 = 0.1$ and, just as in figure 2, $l = 1$ and $l' = 2$.

Finally, figure 4 shows a three-dimensional plot illustrating the variation of $|\phi|$ with ξ_2 for fixed ξ_1.

In the module $\tilde{M}$ defined in (4.19), the coefficients w_l are bicomplex numbers. If they are restricted to elements of $\mathbb{C}(\mathrm{i}_1)$, then the set of linear combinations makes up

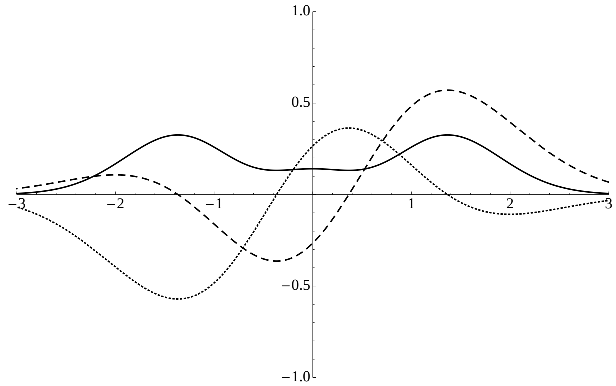

Figure 2: Eigenfunction (4.23) for $\xi_1 = 1 = \xi_2$ and $l = 1$, $l' = 2$

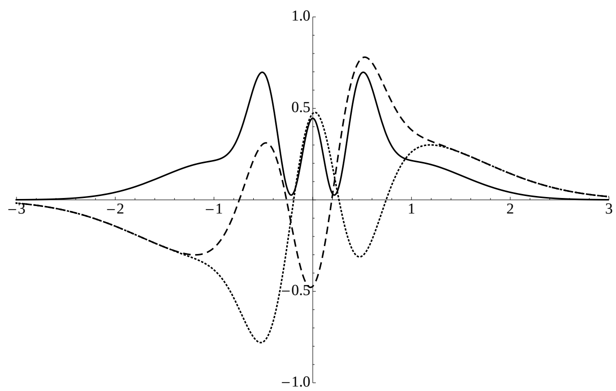

Figure 3: Eigenfunction (4.23) for $\xi_1 = 1$, $\xi_2 = 0.1$ and $l = 1$, $l' = 2$

a vector space $\tilde{V}$, isomorphic to the space V defined after (3.35). The space $\tilde{V}$ is not restricted to standard Hermite polynomials but contains all the hyperbolic ones.

We should note that the module M as we defined it does not have a property of completeness. Indeed it is made up of all finite linear combinations of basis kets. Cauchy sequences of such kets are not expected to converge to an element of the set. It was shown in [11] that the concept of Hilbert space can be adapted to finite-dimensional free $\mathbb{T}$-modules. We believe that by making use of the subspace V of M, the concept of Hilbert space can be extended to infinite-dimensional free $\mathbb{T}$-modules like the one constructed here and based on the bicomplex harmonic oscillator eigenfunctions. We intend to investigate this in the future.

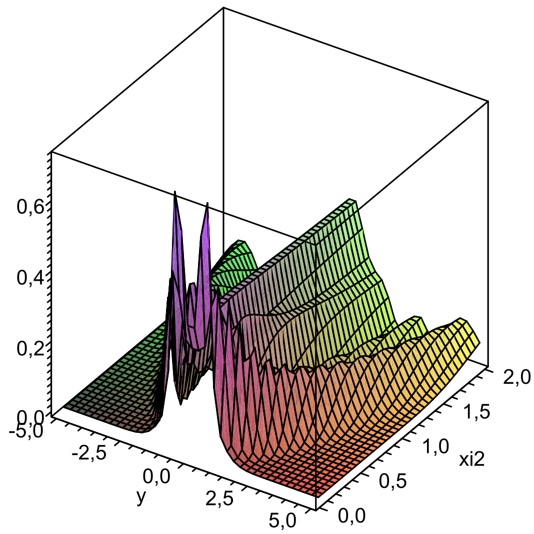

Figure 4: The function $|\phi|^2$, with ϕ given in (4.23), for $l = 0$, $l' = 6$, $\xi_1 = 1$ and $0 < \xi_2 \leq 2$

Acknowledgments

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3.2 Deuxième article

Le second article [1] représente essentiellement une synthèse de résultats et d'outils mathématiques qui ont été développés en grande partie durant la résolution du problème de l'oscillateur harmonique quantique bicomplexe [3]. Les résultats sont divisés principalement en deux parties, ceux indépendants de tout produit scalaire, et ceux nécessitant l'utilisation de celui-ci.

La première partie traite essentiellement d'algèbre linéaire. Un certain nombre de résultats sur les matrices bicomplexes y sont développés comme entre autres la définition du déterminant et de l'inverse. Il est également question de modules bicomplexes de dimension finie, de bases ainsi que d'opérateurs agissant dans ces modules bicomplexes. En particulier, il est démontré que toutes les bases d'un module bicomplexe de dimension finie ont la même dimension et que ces bases sont reliées par des matrices bicomplexes non singulières.

La seconde partie de l'article commence par la définition du produit scalaire bicomplexe. À noter qu'un tel produit scalaire avait déjà été introduit dans [46], comme cas particulier de celui donné ici. À partir de ce produit scalaire bicomplexe, l'espace de Hilbert bicomplexe est défini. La question de l'orthogonalisation ainsi que de la normalisation des éléments d'un module bicomplexe est traitée. Un certain nombre de résultats sur les opérateurs hermitiques bicomplexes sont exposés dont le théorème spectral bicomplexe.

L'article se termine avec quelques résultats à propos des opérateurs unitaires ainsi que la construction de l'opérateur d'évolution en mécanique quantique bicomplexe.

La première version de l'article a été rédigée essentiellement par les trois auteurs. RGL a majoritairement contribué aux sections 3.2, 3.3, 4.4, 5.1, 5.2 et 5.3, LM aux sections 3.2, 3.3, 4.3, 5.2 et 5.3, et DR aux sections 3.1, 3.2, 4.1, 4.2 et 4.3. La version soumise a été peaufinée et reformulée en grande partie par LM.

À noter qu'un troisième article [2] a également été soumis mais que celui-ci n'a pas été incorporé au mémoire puisqu'il a été majoritairement écrit par DR. Cependant, cet article explore les résultats obtenus en [1] dans le cadre plus vaste des modules bicomplexes de dimension infinie.

Finite-Dimensional Bicomplex Hilbert Spaces

Raphaël Gervais Lavoie, Louis Marchildon and Dominic Rochon

Abstract. This paper is a detailed study of finite-dimensional modules defined on bicomplex numbers. A number of results are proved on bicomplex square matrices, linear operators, orthogonal bases, self-adjoint operators and Hilbert spaces, including the spectral decomposition theorem. Applications to concepts relevant to quantum mechanics, like the evolution operator, are pointed out.

Mathematics Subject Classification (2010). 16D10,30G35,46C05,46C50.

Keywords. Bicomplex numbers, bicomplex quantum mechanics, generalized quantum mechanics, Hilbert spaces, bicomplex matrix, bicomplex linear algebra, generalized linear algebra.

1. Introduction

Bicomplex numbers [1], just like quaternions, are a generalization of complex numbers by means of entities specified by four real numbers. These two number systems, however, are different in two important ways: quaternions, which form a division algebra, are noncommutative, whereas bicomplex numbers are commutative but do not form a division algebra.

Division algebras do not have zero divisors, that is, nonzero elements whose product is zero. Many believe that any attempt to generalize quantum mechanics to number systems other than complex numbers should retain the division algebra property. Indeed considerable work has been done over the years on quaternionic quantum mechanics [2].

In the past few years, however, it was pointed out that several features of quantum mechanics can be generalized to bicomplex numbers. A generalization of Schrödinger's equation for a particle in one dimension was

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proposed [3], and self-adjoint operators were defined on finite-dimensional bicomplex Hilbert spaces [4]. Eigenvalues and eigenfunctions of the bicomplex analogue of the quantum harmonic oscillator Hamiltonian were obtained in full generality [5].

The perspective of generalizing quantum mechanics to bicomplex numbers motivates us in developing further mathematical tools related to finite-dimensional bicomplex Hilbert spaces and operators acting on them. After a brief review of bicomplex numbers and modules in Section 2, we devote Section 3 to a number of results in linear algebra that do not depend on the introduction of a scalar product. Basic properties of bicomplex square matrices are obtained and theorems are proved on bases, idempotent projections and the representation of linear operators. In Section 4 we define the bicomplex scalar product and derive a number of results on Hilbert spaces, orthogonalization and self-adjoint operators, including the spectral decomposition theorem. Section 5 is devoted to applications to unitary operators, functions of operators and the quantum evolution operator. We conclude in Section 6.

2. Basic Notions

This section summarizes known properties of bicomplex numbers and modules, on which the bulk of this paper is based. Proofs and additional results can be found in [1, 3, 4, 6].

2.1. Bicomplex Numbers

2.1.1. Definition. The set $\mathbb{T}$ of *bicomplex numbers* is defined as

$$\mathbb{T} := \{w = z_1 + z_2\mathbf{i_2} \mid z_1, z_2 \in \mathbb{C}(\mathbf{i_1})\}, \tag{2.1}$$

where $\mathbf{i_1}$, $\mathbf{i_2}$ and $\mathbf{j}$ are imaginary and hyperbolic units such that $\mathbf{i_1}^2 = -1 = \mathbf{i_2}^2$ and $\mathbf{j}^2 = 1$. The product of units is commutative and is defined as

$$\mathbf{i_1}\mathbf{i_2} = \mathbf{j}, \qquad \mathbf{i_1}\mathbf{j} = -\mathbf{i_2}, \qquad \mathbf{i_2}\mathbf{j} = -\mathbf{i_1}. \tag{2.2}$$

With the addition and multiplication of two bicomplex numbers defined in the obvious way, the set $\mathbb{T}$ makes up a commutative ring.

Three important subsets of $\mathbb{T}$ can be specified as

$$\mathbb{C}(\mathbf{i_k}) := \{x + y\mathbf{i_k} \mid x, y \in \mathbb{R}\}, \qquad k = 1, 2; \tag{2.3}$$

$$\mathbb{D} := \{x + y\mathbf{j} \mid x, y \in \mathbb{R}\}. \tag{2.4}$$

Each of the sets $\mathbb{C}(\mathbf{i_k})$ is isomorphic to the field of complex numbers, while $\mathbb{D}$ is the set of so-called *hyperbolic numbers*.

2.1.2. Conjugation and Moduli. Three kinds of conjugation can be defined on bicomplex numbers. With w specified as in (2.1) and the bar ($\bar{\ }$) denoting complex conjugation in $\mathbb{C}(\mathbf{i_1})$, we define

$$w^{\dagger_1} := \bar{z}_1 + \bar{z}_2\mathbf{i_2}, \qquad w^{\dagger_2} := z_1 - z_2\mathbf{i_2}, \qquad w^{\dagger_3} := \bar{z}_1 - \bar{z}_2\mathbf{i_2}. \tag{2.5}$$

It is easy to check that each conjugation has the following properties:

$$(s+t)^{\dagger_k} = s^{\dagger_k} + t^{\dagger_k}, \qquad \left(s^{\dagger_k}\right)^{\dagger_k} = s, \qquad (s\cdot t)^{\dagger_k} = s^{\dagger_k}\cdot t^{\dagger_k}. \tag{2.6}$$

Here $s, t \in \mathbb{T}$ and $k = 1, 2, 3$.

With each kind of conjugation, one can define a specific bicomplex modulus as

$$|w|^2_{\mathbf{i_1}} := w\cdot w^{\dagger_2} = z_1^2 + z_2^2 \ \in \mathbb{C}(\mathbf{i_1}), \tag{2.7a}$$

$$|w|^2_{\mathbf{i_2}} := w\cdot w^{\dagger_1} = \left(|z_1|^2 - |z_2|^2\right) + 2\,\mathrm{Re}(z_1\bar{z}_2)\mathbf{i_2} \ \in \mathbb{C}(\mathbf{i_2}), \tag{2.7b}$$

$$|w|^2_{\mathbf{j}} := w\cdot w^{\dagger_3} = \left(|z_1|^2 + |z_2|^2\right) - 2\,\mathrm{Im}(z_1\bar{z}_2)\mathbf{j} \ \in \mathbb{D}. \tag{2.7c}$$

It can be shown that $|s\cdot t|^2_k = |s|^2_k \cdot |t|^2_k$, where $k = \mathbf{i_1}, \mathbf{i_2}$ or $\mathbf{j}$.

In this paper we will often use the Euclidean $\mathbb{R}^4$ norm defined as

$$|w| := \sqrt{|z_1|^2 + |z_2|^2} = \sqrt{\mathrm{Re}(|w|^2_{\mathbf{j}})}\ . \tag{2.8}$$

Clearly, this norm maps $\mathbb{T}$ into $\mathbb{R}$. We have $|w| \geq 0$, and $|w| = 0$ if and only if $w = 0$. Moreover [1], for all $s, t \in \mathbb{T}$,

$$|s+t| \leq |s| + |t|, \qquad |s\cdot t| \leq \sqrt{2}|s|\cdot|t|. \tag{2.9}$$

2.1.3. Idempotent Basis. Bicomplex algebra is considerably simplified by the introduction of two bicomplex numbers $\mathbf{e_1}$ and $\mathbf{e_2}$ defined as

$$\mathbf{e_1} := \frac{1+\mathbf{j}}{2}, \qquad \mathbf{e_2} := \frac{1-\mathbf{j}}{2}. \tag{2.10}$$

In fact $\mathbf{e_1}$ and $\mathbf{e_2}$ are hyperbolic numbers. They make up the so-called *idempotent basis* of the bicomplex numbers. One easily checks that $(k = 1, 2)$

$$\mathbf{e_1}^2 = \mathbf{e_1}, \quad \mathbf{e_2}^2 = \mathbf{e_2}, \quad \mathbf{e_1} + \mathbf{e_2} = 1, \quad \mathbf{e_k}^{\dagger_3} = \mathbf{e_k}, \quad \mathbf{e_1}\mathbf{e_2} = 0. \tag{2.11}$$

Any bicomplex number w can be written uniquely as

$$w = z_1 + z_2\mathbf{i_2} = z_{\hat{1}}\mathbf{e_1} + z_{\hat{2}}\mathbf{e_2}, \tag{2.12}$$

where

$$z_{\hat{1}} = z_1 - z_2\mathbf{i_1} \quad \text{and} \quad z_{\hat{2}} = z_1 + z_2\mathbf{i_1} \tag{2.13}$$

both belong to $\mathbb{C}(\mathbf{i_1})$. The caret notation ($\hat{1}$ and $\hat{2}$) will be used systematically in connection with idempotent decompositions, with the purpose of easily distinguishing different types of indices. As a consequence of (2.11) and (2.12), one can check that if $\sqrt[n]{z_{\hat{1}}}$ is an nth root of $z_{\hat{1}}$ and $\sqrt[n]{z_{\hat{2}}}$ is an nth root of $z_{\hat{2}}$, then $\sqrt[n]{z_{\hat{1}}}\,\mathbf{e_1} + \sqrt[n]{z_{\hat{2}}}\,\mathbf{e_2}$ is an nth root of w.

The uniqueness of the idempotent decomposition allows the introduction of two projection operators as

$$P_1 : w \in \mathbb{T} \mapsto z_{\hat{1}} \in \mathbb{C}(\mathbf{i_1}), \tag{2.14}$$

$$P_2 : w \in \mathbb{T} \mapsto z_{\hat{2}} \in \mathbb{C}(\mathbf{i_1}). \tag{2.15}$$

The P_k $(k = 1, 2)$ satisfy

$$[P_k]^2 = P_k, \qquad P_1\mathbf{e_1} + P_2\mathbf{e_2} = \mathbf{Id}, \tag{2.16}$$

and, for $s, t \in \mathbb{T}$,

$$P_k(s+t) = P_k(s) + P_k(t), \qquad P_k(s \cdot t) = P_k(s) \cdot P_k(t). \tag{2.17}$$

The product of two bicomplex numbers w and w' can be written in the idempotent basis as

$$w \cdot w' = (z_{\hat{1}}\mathbf{e_1} + z_{\hat{2}}\mathbf{e_2}) \cdot (z'_{\hat{1}}\mathbf{e_1} + z'_{\hat{2}}\mathbf{e_2}) = z_{\hat{1}}z'_{\hat{1}}\mathbf{e_1} + z_{\hat{2}}z'_{\hat{2}}\mathbf{e_2}. \tag{2.18}$$

Since 1 is uniquely decomposed as $\mathbf{e_1} + \mathbf{e_2}$, we can see that $w \cdot w' = 1$ if and only if $z_{\hat{1}}z'_{\hat{1}} = 1 = z_{\hat{2}}z'_{\hat{2}}$. Thus w has an inverse if and only if $z_{\hat{1}} \neq 0 \neq z_{\hat{2}}$, and the inverse w^{-1} is then equal to $(z_{\hat{1}})^{-1}\mathbf{e_1} + (z_{\hat{2}})^{-1}\mathbf{e_2}$. A nonzero w that does not have an inverse has the property that either $z_{\hat{1}} = 0$ or $z_{\hat{2}} = 0$, and such a w is a divisor of zero. Zero divisors make up the so-called *null cone* $\mathcal{NC}$. That terminology comes from the fact that when w is written as in (2.1), zero divisors are such that $z_1^2 + z_2^2 = 0$.

Any hyperbolic number can be written in the idempotent basis as $x_{\hat{1}}\mathbf{e_1} + x_{\hat{2}}\mathbf{e_2}$, with $x_{\hat{1}}$ and $x_{\hat{2}}$ in $\mathbb{R}$. We define the set $\mathbb{D}^+$ of positive hyperbolic numbers as

$$\mathbb{D}^+ := \{x_{\hat{1}}\mathbf{e_1} + x_{\hat{2}}\mathbf{e_2} \mid x_{\hat{1}}, x_{\hat{2}} \geq 0\}. \tag{2.19}$$

Since $w^{\dagger_3} = \bar{z}_{\hat{1}}\mathbf{e_1} + \bar{z}_{\hat{2}}\mathbf{e_2}$, it is clear that $w \cdot w^{\dagger_3} \in \mathbb{D}^+$ for any w in $\mathbb{T}$.

2.2. $\mathbb{T}$-Modules and Linear Operators

Bicomplex numbers make up a commutative ring. What vector spaces are to fields, modules are to rings. A module defined over the ring $\mathbb{T}$ of bicomplex numbers will be called a $\mathbb{T}$-*module*.

Definition 2.1. A *basis* of a $\mathbb{T}$-module is a set of linearly independent elements that generate the module.[1]

A finite-dimensional *free* $\mathbb{T}$-module is a $\mathbb{T}$-module with a finite basis. That is, M is a finite-dimensional free $\mathbb{T}$-module if there exist n linearly independent elements (denoted $|m_l\rangle$) such that any element $|\psi\rangle$ of M can be written as

$$|\psi\rangle = \sum_{l=1}^{n} w_l |m_l\rangle, \tag{2.20}$$

with $w_l \in \mathbb{T}$. We have used Dirac's notation for elements of M which, following [4], we will call *kets*.

An important subset V of M is the set of all kets for which all w_l in (2.20) belong to $\mathbb{C}(\mathbf{i_1})$. In other words, V is the set of all $|\psi\rangle$ so that

$$|\psi\rangle = \sum_{l=1}^{n} x_l |m_l\rangle, \qquad x_l \in \mathbb{C}(\mathbf{i_1}). \tag{2.21}$$

[1] The term "basis" here should not be confused with the same word appearing in "idempotent basis". Elements of the former belong to the module, while elements of the latter are (bicomplex) numbers.

It was shown in [4] that V is a vector space over the complex numbers, and that any $|\psi\rangle \in \mathbb{T}$ can be decomposed uniquely as

$$|\psi\rangle = \mathbf{e_1}|\psi\rangle_{\widehat{1}} + \mathbf{e_2}|\psi\rangle_{\widehat{2}} = \mathbf{e_1}P_1(|\psi\rangle) + \mathbf{e_2}P_2(|\psi\rangle). \tag{2.22}$$

Here $|\psi\rangle_{\widehat{k}} \in V$ and P_k is a projector from M to V $(k = 1, 2)$. One can show that ket projectors and idempotent-basis projectors (denoted with the same symbol) satisfy

$$P_k(s|\psi\rangle + t|\phi\rangle) = P_k(s)P_k(|\psi\rangle) + P_k(t)P_k(|\phi\rangle). \tag{2.23}$$

A *bicomplex linear operator* A is a mapping from M to M such that, for any $s, t \in \mathbb{T}$ and any $|\psi\rangle, |\phi\rangle \in M$,

$$A(s|\psi\rangle + t|\phi\rangle) = sA|\psi\rangle + tA|\phi\rangle. \tag{2.24}$$

A bicomplex linear operator A can always be written as

$$A = \mathbf{e_1}A_{\widehat{1}} + \mathbf{e_2}A_{\widehat{2}} = \mathbf{e_1}P_1(A) + \mathbf{e_2}P_2(A), \tag{2.25}$$

where $P_k(A)$ $(k = 1, 2)$ was defined in [4] as

$$P_k(A)|\psi\rangle = P_k(A|\psi\rangle) \qquad \forall|\psi\rangle \in M. \tag{2.26}$$

Clearly one can write

$$A|\psi\rangle = \mathbf{e_1}A_{\widehat{1}}|\psi\rangle_{\widehat{1}} + \mathbf{e_2}A_{\widehat{2}}|\psi\rangle_{\widehat{2}}. \tag{2.27}$$

Definition 2.2. A ket $|\psi\rangle$ belongs to the null cone if either $|\psi\rangle_{\widehat{1}} = 0$ or $|\psi\rangle_{\widehat{2}} = 0$. A linear operator A belongs to the null cone if either $A_{\widehat{1}} = 0$ or $A_{\widehat{2}} = 0$.

The following definition adapts to modules the concepts of eigenvector and eigenvalue, most useful in vector space theory.

Definition 2.3. Let $A : M \to M$ be a bicomplex linear operator and let

$$A|\psi\rangle = \lambda|\psi\rangle, \tag{2.28}$$

with $\lambda \in \mathbb{T}$ and $|\psi\rangle \in M$ such that $|\psi\rangle \notin \mathcal{NC}$. Then λ is called an *eigenvalue* of A and $|\psi\rangle$ is called an *eigenket* of A.

Just as eigenvectors are normally restricted to nonzero vectors, we have restricted eigenkets to kets that are not in the null cone. One can show that the eigenket equation (2.28) is equivalent to the following system of two eigenvector equations $(k = 1, 2)$:

$$A_{\widehat{k}}|\psi\rangle_{\widehat{k}} = \lambda_{\widehat{k}}|\psi\rangle_{\widehat{k}}\ . \tag{2.29}$$

Here $\lambda = \mathbf{e_1}\lambda_{\widehat{1}} + \mathbf{e_2}\lambda_{\widehat{2}}$, and $|\psi\rangle_{\widehat{k}}$ and $A_{\widehat{k}}$ have been defined before. For a complete treatment of module theory, see [7, 8].

3. Bicomplex Linear Algebra

3.1. Square Matrices

A bicomplex $n \times n$ square matrix A is an array of n^2 bicomplex numbers A_{ij}. Since each A_{ij} can be expressed in the idempotent basis, it is clear that

$$A = \mathbf{e_1} A_{\widehat{1}} + \mathbf{e_2} A_{\widehat{2}}, \tag{3.1}$$

where $A_{\widehat{1}}$ and $A_{\widehat{2}}$ are two complex $n \times n$ matrices (i.e. with elements in $\mathbb{C}(\mathbf{i_1})$).

Theorem 3.1. *Let $A = \mathbf{e_1} A_{\widehat{1}} + \mathbf{e_2} A_{\widehat{2}}$ be an $n \times n$ bicomplex matrix. Then* $\det(A) = \mathbf{e_1} \det\left(A_{\widehat{1}}\right) + \mathbf{e_2} \det\left(A_{\widehat{2}}\right)$.

Proof. We follow the proof given in [9]. Let $\{C_i\}$ be the set of columns of A, so that $A \equiv (C_1, C_2, \ldots, C_n)$. Let the ith column be such that $C_i = \alpha C_i' + \beta C_i''$, with $\alpha, \beta \in \mathbb{T}$. Since matrix elements belong to a commutative ring, the determinant function satisfies

$$\begin{aligned} &\det(C_1, C_2, \ldots, \alpha C_i' + \beta C_i'', \ldots, C_n) \\ &= \alpha \det(C_1, C_2, \ldots, C_i', \ldots, C_n) + \beta \det(C_1, C_2, \ldots, C_i'', \ldots, C_n). \end{aligned}$$

With a bicomplex matrix, we can write $C_1 = \mathbf{e_1} C_1' + \mathbf{e_2} C_1''$, where columns C_1' and C_2' have entries in $\mathbb{C}(\mathbf{i_1})$. Hence

$$\det(C_1, \ldots, C_n) = \mathbf{e_1} \det(C_1', \ldots, C_n) + \mathbf{e_2} \det(C_1'', \ldots, C_n).$$

Applying this successively to all columns, we find that

$$\det(C_1, \ldots, C_n) = \mathbf{e_1} \det(C_1', \ldots, C_n') + \mathbf{e_2} \det(C_1'', \ldots, C_n''),$$

which is our result. □

From Theorem 3.1 we immediately see that $\det(A) = 0$ if and only if $\det(A_{\widehat{1}}) = 0 = \det(A_{\widehat{2}})$, and that $\det(A)$ is in the null cone if and only if $\det(A_{\widehat{1}}) = 0$ or $\det(A_{\widehat{2}}) = 0$. Moreover, one can easily prove that for any bicomplex square matrices A and B, $\det\left(A^T\right) = \det(A)$ (the superscript T denotes the transpose) and $\det(AB) = \det(A)\det(B)$.

Definition 3.2. A bicomplex square matrix is *singular* if its determinant is in the null cone.

Theorem 3.3. *The inverse A^{-1} of a bicomplex square matrix A exists if and only if A is not singular, and then A^{-1} is given by $\mathbf{e_1}(A_{\widehat{1}})^{-1} + \mathbf{e_2}(A_{\widehat{2}})^{-1}$.*

Proof. If A is not singular, then $\det(A_{\widehat{1}}) \neq 0$ and $\det(A_{\widehat{2}}) \neq 0$, so that $(A_{\widehat{1}})^{-1}$ and $(A_{\widehat{2}})^{-1}$ both exist. But then

$$\left(\mathbf{e_1}(A_{\widehat{1}})^{-1} + \mathbf{e_2}(A_{\widehat{2}})^{-1}\right)\left(\mathbf{e_1} A_{\widehat{1}} + \mathbf{e_2} A_{\widehat{2}}\right) = \mathbf{e_1} I + \mathbf{e_2} I = I.$$

Conversely, if A^{-1} exists, then $A^{-1}A = I$. Hence

$$1 = \det(I) = \det\left(A^{-1}A\right) = \det\left(A^{-1}\right)\det(A),$$

from which we deduce that $\det(A)$ is not in the null cone, and therefore that A is not singular. □

Note that although we wrote A^{-1} as a left inverse, we could have written it just as well as a right inverse, and both inverses coincide.

3.2. Free $\mathbb{T}$-Modules and Bases

Throughout this section, M will denote an n-dimensional free $\mathbb{T}$-module and $\{|m_l\rangle\}$ will denote a basis of M. Any element $|\psi\rangle$ of M can be expressed as in (2.20).

In a vector space, any nonzero vector can be part of a basis. Not so for $\mathbb{T}$-modules.

Theorem 3.4. *No basis element of a free $\mathbb{T}$-module can belong to the null cone.*

Proof. Let $|s_p\rangle$ be an element of a basis of M (not necessarily the $\{|m_l\rangle\}$ basis). By (2.22) we can write

$$|s_p\rangle = \mathbf{e_1}|s_p\rangle_{\widehat{1}} + \mathbf{e_2}|s_p\rangle_{\widehat{2}}. \tag{3.2}$$

Suppose that $|s_p\rangle$ belongs to the null cone. Then either $|s_p\rangle_{\widehat{1}} = 0$ or $|s_p\rangle_{\widehat{2}} = 0$. In the first case $\mathbf{e_1}|s_p\rangle = 0$ and in the second case $\mathbf{e_2}|s_p\rangle = 0$. Both these equations contradict linear independence. □

We now define two important subsets of M.

Definition 3.5. For $k = 1, 2$, $V_k := \{\mathbf{e_k} \sum_{l=1}^{n} x_l |m_l\rangle \mid x_l \in \mathbb{C}(\mathbf{i_1})\}$. Or succinctly, $V_k := \mathbf{e_k} V$.

Clearly, V_k is an n-dimensional vector space over $\mathbb{C}(\mathbf{i_1})$, isomorphic to V and with $\mathbf{e_k}|m_l\rangle$ as basis elements. All three vector spaces V, V_1 and V_2 are useful. Many results proved in [4] used V in a crucial way, while the computation of harmonic oscillator eigenvalues and eigenkets in [5] was based on infinite-dimensional analogues of V_1 and V_2.

Although V depends on the choice of basis $\{|m_l\rangle\}$, V_1 and V_2 do not. This comes from the fact that any element of V_1 (for instance) can be written as $\mathbf{e_1}|\psi\rangle$, with $|\psi\rangle$ in M. Clearly, this makes no reference to any specific basis.

The module M, defined over the ring $\mathbb{T}$, has n dimensions. We now show that the set of elements of M can also be viewed as a $2n$-dimensional vector space over $\mathbb{C}(\mathbf{i_1})$, which we shall call M'. To see this, we write in the idempotent basis the coefficients w_l of a general element of M. Making use of (2.12) and (2.20), we get

$$|\psi\rangle = \sum_{l=1}^{n} (\mathbf{e_1} w_{l\widehat{1}} + \mathbf{e_2} w_{l\widehat{2}})|m_l\rangle = \sum_{l=1}^{n} w_{l\widehat{1}} \mathbf{e_1}|m_l\rangle + \sum_{l=1}^{n} w_{l\widehat{2}} \mathbf{e_2}|m_l\rangle. \tag{3.3}$$

It is not difficult to show that the $2n$ elements $\mathbf{e_1}|m_l\rangle$ and $\mathbf{e_2}|m_l\rangle$ $(l = 1 \dots n)$ are linearly independent over $\mathbb{C}(\mathbf{i_1})$. This proves our claim and, moreover, proves

Theorem 3.6. $M' = V_1 \oplus V_2$.

It is well known that all bases of a finite-dimensional vector space have the same number of elements. This, however, is not true in general for modules [7]. But for $\mathbb{T}$-modules we have

Theorem 3.7. *Let M be a finite-dimensional free $\mathbb{T}$-module. Then all bases of M have the same number of elements.*

Proof. Let $\{|m_l\rangle, l = 1 \ldots n\}$ and $\{|s_p\rangle, p = 1 \ldots m\}$ be two bases of M. We can write

$$\begin{aligned} M &= \left\{ \sum_{p=1}^{m} w_p |s_p\rangle \mid w_p \in \mathbb{T} \right\} \\ &= \left\{ \sum_{p=1}^{m} (P_1(w_p)\mathbf{e_1} + P_2(w_p)\mathbf{e_2}) |s_p\rangle \mid w_p \in \mathbb{T} \right\}, \end{aligned}$$

where, as usual, P_1 and P_2 are defined with respect to the $|m_l\rangle$. Since

$$(P_1(w_p)\mathbf{e_1} + P_2(w_p)\mathbf{e_2})\,|s_p\rangle = P_1(w_p)\mathbf{e_1}|s_p\rangle + P_2(w_p)\mathbf{e_2}|s_p\rangle$$

and $P_k(w_p) \in \mathbb{C}(\mathbf{i_1})$ for $k = 1, 2$, we see that $\{\mathbf{e_k}|s_p\rangle \mid p = 1 \ldots m\}$ is a basis of V_k. But

$$\dim(V_1) = \dim(V_2) = \dim(V) = n,$$

whence $m = n$. □

With the projections P_k defined with respect to the $|m_l\rangle$, it is obvious that $P_k(|m_l\rangle) = |m_l\rangle$ $(k = 1, 2)$. This is a direct consequence of the identity $|m_l\rangle = \mathbf{e_1}|m_l\rangle + \mathbf{e_2}|m_l\rangle$. Hence $\{P_k(|m_l\rangle) \mid l = 1 \ldots n\}$ is a basis of V. It turns out that the projection of any basis of M is a basis of V.

Theorem 3.8. *Let P_1 and P_2 be the projections defined with respect to a basis $\{|m_l\rangle\}$ of M, and let V be the associated vector space. If $\{|s_l\rangle\}$ is another basis of M, then $\{P_1(|s_l\rangle)\}$ and $\{P_2(|s_l\rangle)\}$ are bases of V.*

Proof. We give the proof for P_1, the one for P_2 being similar. We first show that the $P_1(|s_l\rangle)$ are linearly independent, and then that they generate V.

Let $\alpha_l \in \mathbb{C}(\mathbf{i_1})$ for $l = 1 \ldots n$ and let

$$\sum_{i=1}^{n} \alpha_l P_1(|s_l\rangle) = 0.$$

For $l = 1 \ldots n$, define $c_l := \mathbf{e_1}\alpha_l + \mathbf{e_2} \cdot 0$. Making use of (2.23), it is easy to see that $\sum_{l=1}^{n} c_l |s_l\rangle = 0$, for

$$\begin{aligned} P_1\left(\sum_{l=1}^{n} c_l|s_l\rangle\right) &= \sum_{l=1}^{n} P_1(c_l)P_1(|s_l\rangle) = \sum_{l=1}^{n} \alpha_l P_1(|s_l\rangle) = 0, \\ P_2\left(\sum_{l=1}^{n} c_l|s_l\rangle\right) &= \sum_{l=1}^{n} P_2(c_l)P_2(|s_l\rangle) = \sum_{l=1}^{n} 0 \cdot P_2(|s_l\rangle) = 0. \end{aligned}$$

The linear independence (in M) of $\{|s_l\rangle\}$ implies that $\forall l$, $c_l = 0$ and therefore $\alpha_l = 0$.

To show that the $P_1(|s_l\rangle)$ generate V, let $|\psi_1\rangle \in V$ and consider the ket

$$|\psi\rangle := \mathbf{e_1}|\psi_1\rangle + \mathbf{e_2} \cdot 0 \in M.$$

Since the (bicomplex) span of $\{|s_l\rangle\}$ is M, there exist $d_l \in \mathbb{T}$ such that

$$\sum_{l=1}^{n} d_l|s_l\rangle = |\psi\rangle.$$

Therefore,

$$|\psi_1\rangle = P_1(|\psi\rangle) = P_1\left(\sum_{l=1}^{n} d_l|s_l\rangle\right) = \sum_{l=1}^{n} P_1(d_l)P_1(|s_l\rangle).$$

Thus, the (complex) span of $\{P_1(|s_l\rangle)\}$ is the vector space V and $\{P_1(|s_l\rangle)\}$ is a basis of V. □

Corollary 3.9. *Let $|\psi\rangle$ be in M. If $|\psi\rangle_{\hat{1}}$ ($|\psi\rangle_{\hat{2}}$) vanishes, then the projection P_1 (P_2) of all components of $|\psi\rangle$ in any basis vanishes.*

Proof. Let $\{|s_l\rangle\}$ be any basis of M and let $|\psi\rangle_{\hat{1}} = 0$ (the case with $|\psi\rangle_{\hat{2}}$ is similar). One can write

$$|\psi\rangle = \sum_{l=1}^{n} c_l|s_l\rangle.$$

Making use of (2.22) and (2.23), we get

$$0 = |\psi\rangle_{\hat{1}} = P_1(|\psi\rangle) = \sum_{l=1}^{n} P_1(c_l)P_1(|s_l\rangle).$$

Since the $P_1(|s_l\rangle)$ are linearly independent, we find that $\forall l$, $P_1(c_l) = 0$. □

It is well known that two arbitrary bases of a finite-dimensional vector space are related by a nonsingular matrix, where in that context nonsingular means having nonvanishing determinant. Definition 3.2 of a singular bicomplex matrix (as one whose determinant is in the null cone) leads to the following analogous theorem.

Theorem 3.10. *Any two bases of M are related by a nonsingular matrix.*

Proof. Let $\{|m_l\rangle\}$ and $\{|s_l\rangle\}$ be two bases of M. From Theorem 3.7, we know that both bases have the same dimension n. We can write

$$|m_l\rangle = \sum_{p=1}^{n} L_{pl}|s_p\rangle, \qquad |s_p\rangle = \sum_{j=1}^{n} N_{jp}|m_j\rangle,$$

where L and N are both $n \times n$ bicomplex matrices. But then

$$|m_l\rangle = \sum_{p=1}^{n} L_{pl} \sum_{j=1}^{n} N_{jp}|m_j\rangle = \sum_{j=1}^{n} \left\{\sum_{p=1}^{n} N_{jp}L_{pl}\right\}|m_j\rangle.$$

This means that for any l,

$$\sum_{j=1}^{n} \{\delta_{jl} - (NL)_{jl}\} \, |m_j\rangle = 0.$$

Since the $|m_j\rangle$ are linearly independent, we get that $\delta_{jl} - (NL)_{jl} = 0$ for all l and j, or $NL = I$. Hence L and N are inverses of each other and, by Theorem 3.3, nonsingular. □

3.3. Linear Operators

In this section we first prove a result on the composition of two linear operators, and then establish the equivalence between linear operators and square matrices for bicomplex numbers.

Theorem 3.11. *Let $A, B : M \to M$ be two bicomplex linear operators. Then for $k = 1, 2$,*

1. $P_k(A + B) = P_k(A) + P_k(B)$,
2. $P_k(A \circ B) = P_k(A) \circ P_k(B)$,

where $A \circ B$ denotes the operator that acts on an arbitrary $|\psi\rangle$ as $(A \circ B)\,|\psi\rangle = A\,(B|\psi\rangle)$.

Proof. To prove the first part, we let $|\psi\rangle \in M$ and make use of (2.23) and (2.26). We get

$$\begin{aligned}(P_k(A + B))\,|\psi\rangle &= P_k((A + B)|\psi\rangle) = P_k(A|\psi\rangle + B|\psi\rangle) \\ &= P_k(A|\psi\rangle) + P_k(B|\psi\rangle) = P_k(A)|\psi\rangle + P_k(B)|\psi\rangle \\ &= (P_k(A) + P_k(B))\,|\psi\rangle.\end{aligned}$$

To prove the second part we use (2.25) and (2.26) to get

$$\begin{aligned}(A \circ B)\,|\psi\rangle &= A\,(B|\psi\rangle) \\ &= [\mathbf{e_1} P_1(A) + \mathbf{e_2} P_2(A)] \left\{ [\mathbf{e_1} P_1(B) + \mathbf{e_2} P_2(B)]\,|\psi\rangle \right\} \\ &= [\mathbf{e_1} P_1(A) + \mathbf{e_2} P_2(A)]\,[\mathbf{e_1} P_1(B|\psi\rangle) + \mathbf{e_2} P_2(B|\psi\rangle)] \\ &= \mathbf{e_1} P_1(A) P_1(B|\psi\rangle) + \mathbf{e_2} P_2(A) P_2(B|\psi\rangle).\end{aligned}$$

Applying P_k on both side, we find that $P_k((A \circ B)\,|\psi\rangle) = P_k(A) P_k(B|\psi\rangle)$ or, equivalently, $P_k(A \circ B) = P_k(A) \circ P_k(B)$. □

Theorem 3.12. *The action of a linear bicomplex operator on M can be represented by a bicomplex matrix.*

Proof. Let $A : M \to M$ be a bicomplex linear operator and let $\{|m_l\rangle\}$ be a basis of M. Let $|\psi\rangle$ be in M and let $|\psi'\rangle := A|\psi\rangle$.

Since $\{|m_l\rangle\}$ is a basis of M, the ket $A|m_l\rangle$ can be represented as a linear combination of the $|m_p\rangle$:

$$A|m_l\rangle = \sum_{p=1}^{n} A_{pl} |m_p\rangle.$$

Writing $|\psi\rangle$ as in (2.20) and making use of (2.24), we get

$$|\psi'\rangle = A|\psi\rangle = \sum_{l=1}^{n} w_l A|m_l\rangle = \sum_{l=1}^{n} w_l \sum_{p=1}^{n} A_{pl}|m_p\rangle$$
$$= \sum_{p=1}^{n} \left\{ \sum_{l=1}^{n} A_{pl} w_l \right\} |m_p\rangle.$$

Writing $|\psi'\rangle = \sum_{p=1}^{n} w'_p|m_p\rangle$ and making use of the linear independence of the $|m_p\rangle$, we obtain

$$w'_p = \sum_{l=1}^{n} A_{pl} w_l.$$

The action of A on $|\psi\rangle$ is thus completely determined by the matrix whose elements are the bicomplex numbers A_{pl}. □

Clearly, the matrix associated with a linear operator depends on the basis in which kets are expressed. Given a specific basis, however, it is not difficult to show that the matrix associated with the operator $A \circ B$ is the product of the matrices associated with A and B.

Let two bases $|m_l\rangle$ and $|s_l\rangle$ be related by $|m_l\rangle = \sum_{p=1}^{n} L_{pl}|s_p\rangle$. Let the linear operator A be represented by the matrix A_{pl} in $|m_l\rangle$ and by the matrix $\tilde{A}_{pl}$ in $|s_l\rangle$. Then one can show that

$$A_{ji} = \sum_{p,l=1}^{n} (L^{-1})_{jp} \tilde{A}_{pl} L_{li}. \tag{3.4}$$

Finally, it is not difficult to show that if $A_{\widehat{1}} = 0$, then $A_{pl\widehat{1}} = 0$ for all p and l, in every basis.

4. Bicomplex Hilbert Spaces

4.1. Scalar Product

The bicomplex scalar product was defined in [4] where, as in this paper, the physicists' convention is used for the order of elements in the product.

Definition 4.1. Let M be a finite-dimensional free $\mathbb{T}$-module. Suppose that with each pair $|\psi\rangle$ and $|\phi\rangle$ in M, taken in this order, we associate a bicomplex number $(|\psi\rangle, |\phi\rangle)$ which, $\forall |\chi\rangle \in M$ and $\forall \alpha \in \mathbb{T}$, satisfies

1. $(|\psi\rangle, |\phi\rangle + |\chi\rangle) = (|\psi\rangle, |\phi\rangle) + (|\psi\rangle, |\chi\rangle)$;
2. $(|\psi\rangle, \alpha|\phi\rangle) = \alpha(|\psi\rangle, |\phi\rangle)$;
3. $(|\psi\rangle, |\phi\rangle) = (|\phi\rangle, |\psi\rangle)^{\dagger_3}$;
4. $(|\psi\rangle, |\psi\rangle) = 0$ if and only if $|\psi\rangle = 0$.

Then we say that $(|\psi\rangle, |\phi\rangle)$ is a *bicomplex scalar product*.

Property 3 implies that $(|\psi\rangle, |\psi\rangle) \in \mathbb{D}$. Definition 4.1 is very general. In this paper we shall be a little more restrictive, by requiring the bicomplex scalar product to be hyperbolic positive, that is,

$$(|\psi\rangle, |\psi\rangle) \in \mathbb{D}^+, \qquad \forall |\psi\rangle \in M. \tag{4.1}$$

This may be a more natural generalization of the scalar product on complex vector spaces, where $(|\psi\rangle, |\psi\rangle)$ is never negative.

Definition 4.2. Let $\{|m_l\rangle\}$ be a basis of M and let V be the associated vector space. We say that a scalar product is $\mathbb{C}(\mathbf{i_1})$-*closed* under V if, $\forall |\psi\rangle, |\phi\rangle \in V$, we have $(|\psi\rangle, |\phi\rangle) \in \mathbb{C}(\mathbf{i_1})$.

We note that the property of being $\mathbb{C}(\mathbf{i_1})$-closed is basis-dependent. That is, a scalar product may be $\mathbb{C}(\mathbf{i_1})$-closed under V defined through a basis $\{|m_l\rangle\}$, but not under V' defined through a basis $\{|s_l\rangle\}$. However, one can show that for $k = 1, 2$, the following projection of a bicomplex scalar product:

$$(\cdot, \cdot)_{\widehat{k}} := P_k((\cdot, \cdot)) : M \times M \longrightarrow \mathbb{C}(\mathbf{i_1}) \tag{4.2}$$

is a **standard scalar product** on V_k as well as on V.

Theorem 4.3. *Let $|\psi\rangle, |\phi\rangle \in M$, then*

$$(|\psi\rangle, |\phi\rangle) = \mathbf{e_1}(|\psi\rangle_{\widehat{1}}, |\phi\rangle_{\widehat{1}})_{\widehat{1}} + \mathbf{e_2}(|\psi\rangle_{\widehat{2}}, |\phi\rangle_{\widehat{2}})_{\widehat{2}}\,. \tag{4.3}$$

Proof. Using Theorem 4 of [4], we have

$$\begin{aligned}
(|\psi\rangle, |\phi\rangle) &= (\mathbf{e_1}|\psi\rangle_{\widehat{1}} + \mathbf{e_2}|\psi\rangle_{\widehat{2}}, \mathbf{e_1}|\phi\rangle_{\widehat{1}} + \mathbf{e_2}|\phi\rangle_{\widehat{2}}) \\
&= \mathbf{e_1}\left(|\psi\rangle_{\widehat{1}}, |\phi\rangle_{\widehat{1}}\right) + \mathbf{e_2}\left(|\psi\rangle_{\widehat{2}}, |\phi\rangle_{\widehat{2}}\right) \\
&= \mathbf{e_1}\left\{\mathbf{e_1}P_1\left((|\psi\rangle_{\widehat{1}}, |\phi\rangle_{\widehat{1}})\right) + \mathbf{e_2}P_2\left((|\psi\rangle_{\widehat{1}}, |\phi\rangle_{\widehat{1}})\right)\right\} \\
&\quad + \mathbf{e_2}\left\{\mathbf{e_1}P_1\left((|\psi\rangle_{\widehat{2}}, |\phi\rangle_{\widehat{2}})\right) + \mathbf{e_2}P_2\left((|\psi\rangle_{\widehat{2}}, |\phi\rangle_{\widehat{2}})\right)\right\} \\
&= \mathbf{e_1}P_1\left((|\psi\rangle_{\widehat{1}}, |\phi\rangle_{\widehat{1}})\right) + \mathbf{e_2}P_2\left((|\psi\rangle_{\widehat{2}}, |\phi\rangle_{\widehat{2}})\right) \\
&= \mathbf{e_1}\left(|\psi\rangle_{\widehat{1}}, |\phi\rangle_{\widehat{1}}\right)_{\widehat{1}} + \mathbf{e_2}\left(|\psi\rangle_{\widehat{2}}, |\phi\rangle_{\widehat{2}}\right)_{\widehat{2}}\,.
\end{aligned}$$

□

Theorem 4.3 is true whether the bicomplex scalar product is $\mathbb{C}(\mathbf{i_1})$-closed under V or not. When it is $\mathbb{C}(\mathbf{i_1})$-closed, we have for $k = 1, 2$

$$(|\psi\rangle, |\phi\rangle)_{\widehat{k}} = P_k((|\psi\rangle, |\phi\rangle)) = (|\psi\rangle, |\phi\rangle)\,, \qquad \forall |\psi\rangle, |\phi\rangle \in V. \tag{4.4}$$

Corollary 4.4. *A ket $|\psi\rangle$ is in the null cone if and only if $(|\psi\rangle, |\psi\rangle)$ is in the null none.*

Proof. By theorem 4.3 we have

$$(|\psi\rangle, |\psi\rangle) = \mathbf{e_1}\left(|\psi\rangle_{\widehat{1}}, |\psi\rangle_{\widehat{1}}\right)_{\widehat{1}} + \mathbf{e_2}\left(|\psi\rangle_{\widehat{2}}, |\psi\rangle_{\widehat{2}}\right)_{\widehat{2}}\,. \tag{4.5}$$

If $|\psi\rangle$ is in the null cone, then $|\psi\rangle_{\widehat{k}} = 0$ for $k = 1$ or 2. By (4.5) then, $\mathbf{e_k}\left(|\psi\rangle, |\psi\rangle\right) = 0$.

Conversely, if $(|\psi\rangle, |\psi\rangle)$ is not in the null cone, then by (4.5)

$$\left(|\psi\rangle_{\widehat{k}}, |\psi\rangle_{\widehat{k}}\right)_{\widehat{k}} \neq 0, \qquad k = 1, 2.$$

But then $\big(|\psi\rangle_{\widehat{k}}, |\psi\rangle_{\widehat{k}}\big) \neq 0$, and therefore $|\psi\rangle_{\widehat{k}} \neq 0$ $(k = 1, 2)$. □

4.2. Hilbert Spaces

Theorem 4.5. *Let M be a finite-dimensional free $\mathbb{T}$-module, let $\{|m_l\rangle\}$ be a basis of M and let V be the vector space associated with $\{|m_l\rangle\}$ through* (2.21). *Then for $k = 1, 2$, $(V, (\cdot,\cdot)_{\widehat{k}})$ and $(V_k, (\cdot,\cdot)_{\widehat{k}})$ are complex $(\mathbb{C}(\mathbf{i_1}))$ pre-Hilbert spaces.*

Proof. Since $(\cdot,\cdot)_{\widehat{k}}$ is a standard scalar product when the vector space M' of Theorem 3.6 is restricted to V or V_k, then $(V, (\cdot,\cdot)_{\widehat{k}})$ and $(V_k, (\cdot,\cdot)_{\widehat{k}})$ are complex $(\mathbb{C}(\mathbf{i_1}))$ pre-Hilbert spaces. □

Corollary 4.6. *$(V, (\cdot,\cdot)_{\widehat{k}})$ and $(V_k, (\cdot,\cdot)_{\widehat{k}})$ are complex $(\mathbb{C}(\mathbf{i_1}))$ Hilbert spaces.*

Proof. Theorem 4.5 implies that pre-Hilbert spaces $(V, (\cdot,\cdot)_{\widehat{k}})$ and $(V_k, (\cdot,\cdot)_{\widehat{k}})$ are finite-dimensional normed spaces over $\mathbb{C}(\mathbf{i_1})$. Therefore they are also complete metric spaces [10]. Hence V and V_k are complex $(\mathbb{C}(\mathbf{i_1}))$ Hilbert spaces. □

Let $|\psi_k\rangle$ and $|\phi_k\rangle$ be in V_k for $k = 1, 2$. On the direct sum of the two Hilbert spaces V_1 and V_2, one can define a scalar product as follows:

$$(|\psi_1\rangle \oplus |\psi_2\rangle, |\phi_1\rangle \oplus |\phi_2\rangle) = (|\psi_1\rangle, |\phi_1\rangle)_{\widehat{1}} + (|\psi_2\rangle, |\phi_2\rangle)_{\widehat{2}}\,. \tag{4.6}$$

Then $M' = V_1 \oplus V_2$ is a Hilbert space [11].

From a set-theoretical point of view, M and M' are identical. In this sense we can say, perhaps improperly, that the **module** M can be decomposed into the direct sum of two classical Hilbert spaces, i.e. $M = V_1 \oplus V_2$. Now let us consider the following **norm** on the vector space M':

$$\big|\big||\phi\rangle\big|\big| := \frac{1}{\sqrt{2}}\sqrt{\big(|\phi\rangle_{\widehat{1}}, |\phi\rangle_{\widehat{1}}\big)_{\widehat{1}} + \big(|\phi\rangle_{\widehat{2}}, |\phi\rangle_{\widehat{2}}\big)_{\widehat{2}}}\,. \tag{4.7}$$

Making use of this norm, we can define a metric on M:

$$d(|\phi\rangle, |\psi\rangle) = \big|\big||\phi\rangle - |\psi\rangle\big|\big|. \tag{4.8}$$

With this metric M is **complete**, and therefore a **bicomplex Hilbert space**.

We note that a bicomplex scalar product is **completely characterized** by the two scalar products $(\cdot,\cdot)_{\widehat{k}}$ on V. In fact if $(\cdot,\cdot)_{\widehat{1}}$ and $(\cdot,\cdot)_{\widehat{2}}$ are two arbitrary scalar products on V, then $(\cdot,\cdot)$ defined in (4.3) is a bicomplex scalar product on M.

As a direct application of this decomposition, we obtain the following important result.

Theorem 4.7. *Let $f : M \to \mathbb{T}$ be a linear functional on M. Then there is a unique $|\psi\rangle \in M$ such that $\forall |\phi\rangle$, $f(|\phi\rangle) = (|\psi\rangle, |\phi\rangle)$.*

Proof. We make use of the analogue theorem on V [10, p. 215], with the functional $P_k(f)$ restricted to V. The theorem shows that for each $k = 1, 2$, there is a unique $|\psi_k\rangle \in V$ such that

$$P_k(f)(|\phi\rangle_{\widehat{k}}) = \big(|\psi_k\rangle, |\phi\rangle_{\widehat{k}}\big)_{\widehat{k}}\,.$$

Making use of Theorem 4.3, we find that $|\psi\rangle := \mathbf{e_1}|\psi_1\rangle + \mathbf{e_2}|\psi_2\rangle$ has the desired properties. □

4.3. Orthogonalization

Just like in vector spaces, a basis in M can always be orthogonalized.

Theorem 4.8. *Let M be a finite-dimensional free $\mathbb{T}$-module and let $\{|s_l\rangle\}$ be an arbitrary basis of M. Then one can always find bicomplex linear combinations of the $|s_l\rangle$ which make up an orthogonal basis.*

Proof. Making use of Theorem 3.6 and Corollary 4.6, we see that $M = V_1 \oplus V_2$, with V_k a complex Hilbert space. By Theorem 3.8, $\{\mathbf{e_k}|s_l\rangle_{\hat{k}}\}$ is a basis of V_k $(k = 1, 2)$. Bases in vector spaces can always be orthogonalized. So let $\{\mathbf{e_k}|s'_l\rangle_{\hat{k}}\}$ be an orthogonal basis made up of linear combinations of the $\mathbf{e_k}|s_l\rangle_{\hat{k}}$. For all $l \in \{1 \dots n\}$ and for $p \neq l$, we see that

$$\begin{aligned}&\big(\mathbf{e_1}|s'_l\rangle_{\hat{1}} + \mathbf{e_2}|s'_l\rangle_{\hat{2}}\, ,\, \mathbf{e_1}|s'_l\rangle_{\hat{1}} + \mathbf{e_2}|s'_l\rangle_{\hat{2}}\big)\\ &\qquad = \big(\mathbf{e_1}|s'_l\rangle_{\hat{1}}, \mathbf{e_1}|s'_l\rangle_{\hat{1}}\big) + \big(\mathbf{e_2}|s'_l\rangle_{\hat{2}}, \mathbf{e_2}|s'_l\rangle_{\hat{2}}\big)\end{aligned}$$

is not in the null cone, and that

$$\begin{aligned}&\big(\mathbf{e_1}|s'_l\rangle_{\hat{1}} + \mathbf{e_2}|s'_l\rangle_{\hat{2}}\, ,\, \mathbf{e_1}|s'_p\rangle_{\hat{1}} + \mathbf{e_2}|s'_p\rangle_{\hat{2}}\big)\\ &\qquad = \big(\mathbf{e_1}|s'_l\rangle_{\hat{1}}, \mathbf{e_1}|s'_p\rangle_{\hat{1}}\big) + \big(\mathbf{e_2}|s'_l\rangle_{\hat{2}}, \mathbf{e_2}|s'_p\rangle_{\hat{2}}\big)\end{aligned}$$

vanishes. This shows that the set $\{\mathbf{e_1}|s'_l\rangle_{\hat{1}} + \mathbf{e_2}|s'_l\rangle_{\hat{2}}\}$ is an orthogonal basis of M. □

It is interesting to see explicitly how the Gram-Schmidt orthogonalization process can be applied. Let $\{|m_l\rangle\}$ be a basis of M. We have shown in Theorem 3.4 that no $|m_l\rangle$, and therefore no $(|m_l\rangle, |m_l\rangle)$, can belong to the null cone. Let $|m'_1\rangle = |m_1\rangle$ and let us define

$$|m'_2\rangle = |m_2\rangle - \frac{(|m'_1\rangle, |m_2\rangle)}{(|m'_1\rangle, |m'_1\rangle)}|m'_1\rangle.$$

Clearly, $|m'_2\rangle$ exists and $(|m'_1\rangle, |m'_2\rangle) = 0$. Moreover, $(|m'_2\rangle, |m'_2\rangle)$ is not in the null cone. If it were, we would have for instance (by Corollary 4.4) $\mathbf{e_2}|m'_2\rangle = 0$. But then

$$\begin{aligned}0 &= \mathbf{e_2}\left(|m_2\rangle - \frac{(|m'_1\rangle, |m_2\rangle)}{(|m'_1\rangle, |m'_1\rangle)}|m'_1\rangle\right)\\ &= \mathbf{e_2}|m_2\rangle + \left(-\frac{(|m'_1\rangle, |m_2\rangle)}{(|m'_1\rangle, |m'_1\rangle)}\mathbf{e_2}\right)|m_1\rangle.\end{aligned}$$

This is impossible, since $|m_1\rangle$ and $|m_2\rangle$ are linearly independent. We can verify that $|m'_1\rangle$ and $|m'_2\rangle$ are also linearly independent. Indeed let $w_1|m'_1\rangle + w_2|m'_2\rangle = 0$. Taking the scalar product of this equation with $|m'_l\rangle$ $(l = 1, 2)$, we find that $w_l\,(|m'_l\rangle, |m'_l\rangle) = 0$. Because $|m'_l\rangle$ is not in the null-cone, w_l must vanish.

Now we can make an inductive argument to generate an orthogonal basis. Suppose that we have k linear combination of $|m_1\rangle, \dots, |m_k\rangle$, denoted

$|m'_1\rangle, \ldots, |m'_k\rangle$, that are mutually orthogonal, linearly independent and not in the null cone. Let us define

$$|m'_{k+1}\rangle = |m_{k+1}\rangle - \frac{(|m'_1\rangle, |m_{k+1}\rangle)}{(|m'_1\rangle, |m'_1\rangle)}|m'_1\rangle - \cdots - \frac{(|m'_k\rangle, |m_{k+1}\rangle)}{(|m'_k\rangle, |m'_k\rangle)}|m'_k\rangle.$$

Clearly, $|m'_{k+1}\rangle$ exists. We now show that $|m'_1\rangle, \ldots, |m'_{k+1}\rangle$ are (i) mutually orthogonal, (ii) not in the null cone and (iii) linearly independent.

To prove (i), it is enough to note that $\big(|m'_l\rangle, |m'_{k+1}\rangle\big) = 0$ for $1 \leq l \leq k$. To prove (ii), let's assume (for instance) that $\mathbf{e_2}|m'_{k+1}\rangle = 0$. We then have

$$0 = \mathbf{e_2}|m_{k+1}\rangle - \frac{\mathbf{e_2}\,(|m'_1\rangle, |m_{k+1}\rangle)}{(|m'_1\rangle, |m'_1\rangle)}|m'_1\rangle - \cdots - \frac{\mathbf{e_2}\,(|m'_k\rangle, |m_{k+1}\rangle)}{(|m'_k\rangle, |m'_k\rangle)}|m'_k\rangle.$$

Because the $|m'_l\rangle$ $(l \leq k)$ are linear combinations of the $|m_l\rangle$, this implies that

$$0 = \mathbf{e_2}|m_{k+1}\rangle + \sum_{l=1}^{k} w_l |m_l\rangle,$$

for some coefficients w_l (possibly null). But this equation is impossible because $|m_{k+1}\rangle$ and $|m_l\rangle$ $(l \leq k)$ are linearly independent.

The proof of (iii), that the $|m'_l\rangle$ $(l \leq k+1)$ are linearly independent, can be carried out just like the one that $|m'_1\rangle$ and $|m'_2\rangle$ are. This completes the orthogonalization process.

Going back to the end of Theorem 4.8, we can see that any set like

$$\{\mathbf{e_1}|s'_{l_1}\rangle_{\hat{1}} + \mathbf{e_2}|s'_{l_2}\rangle_{\hat{2}}\}, \tag{4.9}$$

with l_1 not always equal to l_2, will give a new orthogonal basis of M. Following this procedure, it is possible to construct $n!$ different orthogonal bases of M. Of course, there are an infinite number of bases of M.

The following theorem shows that an orthogonal basis can always be orthonormalized.

Theorem 4.9. *Any ket $|\psi\rangle$ not in the null cone can be normalized.*

Proof. Since $(|\psi\rangle, |\psi\rangle) \in \mathbb{D}^+$ and $|\psi\rangle$ is not in the null cone, we can write

$$(|\psi\rangle, |\psi\rangle) = a\mathbf{e_1} + b\mathbf{e_2}, \tag{4.10}$$

with $a > 0$ and $b > 0$. It is easy to check that the ket

$$|\phi\rangle = \left(\frac{1}{\sqrt{a}}\mathbf{e_1} + \frac{1}{\sqrt{b}}\mathbf{e_2}\right)|\psi\rangle$$

satisfies $(|\phi\rangle, |\phi\rangle) = 1$. □

Note that normalization would be impossible if the scalar product were outside $\mathbb{D}^+$, that is, if either a or b were negative.

4.4. Self-Adjoint Operators

In Theorem 4.7 we showed that with finite-dimensional free $\mathbb{T}$-modules, linear functionals are in one-to-one correspondence with kets and act like scalar products. This allows for the introduction of Dirac's bra notation and the alternative writing of the scalar product $(|\psi\rangle, |\phi\rangle)$ as $\langle\psi|\phi\rangle$.

In [4] the bicomplex *adjoint* operator A^* of A was introduced as the unique operator that satisfies

$$(|\psi\rangle, A|\phi\rangle) = (A^*|\psi\rangle, |\phi\rangle), \qquad \forall|\psi\rangle, |\phi\rangle \in M. \tag{4.11}$$

In finite-dimensional free $\mathbb{T}$-modules the adjoint always exists, is linear and satisfies

$$(A^*)^* = A, \quad (sA + tB)^* = s^{\dagger_3}A^* + t^{\dagger_3}B^*, \quad (AB)^* = B^*A^*. \tag{4.12}$$

Moreover,

$$P_k(A)^* = P_k(A^*), \qquad k = 1, 2, \tag{4.13}$$

where $P_k(A)^*$ is the $\mathbb{C}(\mathbf{i_1})$ adjoint on V.

Lemma 4.10. *Let $|\psi\rangle, |\phi\rangle \in M$. Define an operator $|\phi\rangle\langle\psi|$ so that its action on an arbitrary ket $|\chi\rangle$ is given by $(|\phi\rangle\langle\psi|)|\chi\rangle = |\phi\rangle\,(\langle\psi|\chi\rangle)$. Then $|\phi\rangle\langle\psi|$ is a linear operator on M.*

Proof. For any $|\chi_1\rangle$ and $|\chi_2\rangle$ in M and for any α_1 and α_2 in $\mathbb{T}$, we have

$$\begin{aligned}(|\phi\rangle\langle\psi|)\,(\alpha_1|\chi_1\rangle + \alpha_2|\chi_2\rangle) &= |\phi\rangle\,\{\langle\psi|\,(\alpha_1|\chi_1\rangle + \alpha_2|\chi_2\rangle)\} \\ &= |\phi\rangle\,\{\alpha_1\,\langle\psi|\chi_1\rangle + \alpha_2\,\langle\psi|\chi_2\rangle\} \\ &= \alpha_1|\phi\rangle\,(\langle\psi|\chi_1\rangle) + \alpha_2|\phi\rangle\,(\langle\psi|\chi_2\rangle) \\ &= \alpha_1\,(|\phi\rangle\langle\psi|)\,|\chi_1\rangle + \alpha_2\,(|\phi\rangle\langle\psi|)\,|\chi_2\rangle.\end{aligned}$$

Ring commutativity allowed us to move scalars freely around kets. □

Theorem 4.11. *Let $\{|u_l\rangle\}$ be an orthonormal basis of M. Then*

$$\sum_{l=1}^{n} |u_l\rangle\langle u_l| = I.$$

Proof. Since the action of a linear operator is fully determined by its action on elements of a basis, it suffices to show that the equality holds on elements of any basis. Letting the operator on the left-hand side act on $|u_p\rangle$, we have

$$\left(\sum_{l=1}^{n} |u_l\rangle\langle u_l|\right)|u_p\rangle = \sum_{l=1}^{n} |u_l\rangle(\langle u_l|u_p\rangle) = \sum_{l=1}^{n} |u_l\rangle\delta_{lp} = |u_p\rangle.$$ □

Definition 4.12. A bicomplex linear operator H is called *self-adjoint* if $H^* = H$.

Lemma 4.13. *Let $H : M \to M$ be a self-adjoint operator. Then $P_k(H) : V \to V$ $(k = 1, 2)$ is a self-adjoint operator on V.*

Proof. By (4.13), $P_k(H)^* = P_k(H^*) = P_k(H)$. □

Theorem 4.14. *Two eigenkets of a bicomplex self-adjoint operator are orthogonal if the difference of the two eigenvalues is not in $\mathcal{NC}$.*

Proof. Let $H : M \to M$ be a self-adjoint operator and let $|\phi\rangle$ and $|\phi'\rangle$ be two eigenkets of H associated with eigenvalues λ and λ', respectively. Then

$$\begin{aligned} 0 &= (|\phi\rangle, H|\phi'\rangle) - (|\phi'\rangle, H|\phi\rangle)^{\dagger_3} = \lambda' (|\phi\rangle, |\phi'\rangle) - [\lambda (|\phi'\rangle, |\phi\rangle)]^{\dagger_3} \\ &= \lambda' (|\phi\rangle, |\phi'\rangle) - \lambda^{\dagger_3} (|\phi'\rangle, |\phi\rangle)^{\dagger_3} = (\lambda' - \lambda^{\dagger_3}) (|\phi\rangle, |\phi'\rangle) . \end{aligned}$$

Because H is self-adjoint we know, from Theorem 14 of [4], that $\lambda \in \mathbb{D}$. Hence $\lambda^{\dagger_3} = \lambda$ and if $\lambda' - \lambda \notin \mathcal{NC}$, then $(|\phi\rangle, |\phi'\rangle) = 0$. □

With the structure we have now built, we can prove the spectral decomposition theorem for finite-dimensional bicomplex Hilbert spaces.

Theorem 4.15. *Let M be a finite-dimensional free $\mathbb{T}$-module and let $H : M \to M$ be a bicomplex self-adjoint operator. It is always possible to find a set $\{|\phi_l\rangle\}$ of eigenkets of H that make up an orthonormalized basis of M. Moreover, H can be expressed as*

$$H = \sum_{l=1}^{n} \lambda_l |\phi_l\rangle\langle\phi_l|, \tag{4.14}$$

where λ_l is the eigenvalue of H associated with the eigenket $|\phi_l\rangle$.

Proof. We first remark that the classical spectral decomposition theorem holds for the self-adjoint operator $P_k(H) = H_{\widehat{k}}$, restricted to V $(k = 1, 2)$. So let $\{|\phi_l\rangle_{\widehat{1}}\}$ and $\{|\phi_l\rangle_{\widehat{2}}\}$ be orthonormal sets of eigenvectors of $H_{\widehat{1}}$ and $H_{\widehat{2}}$, respectively. They make up orthonormal bases of V with respect to the scalar products $(\cdot,\cdot)_{\widehat{1}}$ and $(\cdot,\cdot)_{\widehat{2}}$. Letting $|\phi_l\rangle := \mathbf{e_1}|\phi_l\rangle_{\widehat{1}} + \mathbf{e_2}|\phi_l\rangle_{\widehat{2}}$, we can see that $\{|\phi_l\rangle\}$ is an orthonormal basis of M. Let λ_l be the eigenvalue of H associated with $|\phi_l\rangle$, so that $H|\phi_l\rangle = \lambda_l|\phi_l\rangle$. To show that (4.14) holds, it is enough to show that the right-hand side of (4.14) acts on basis kets like H. But

$$\left[\sum_{l=1}^{n} \lambda_l |\phi_l\rangle\langle\phi_l|\right] |\phi_p\rangle = \sum_{l=1}^{n} \lambda_l |\phi_l\rangle(\langle\phi_l|\phi_p\rangle) = \sum_{l=1}^{n} \lambda_l \delta_{lp} |\phi_l\rangle = \lambda_p |\phi_p\rangle.$$ □

5. Applications

As an application of the results obtained in the previous sections, we will develop the bicomplex version of the quantum-mechanical evolution operator. To do this, we first need to define bicomplex unitary operators as well as functions of a bicomplex operator.

5.1. Unitary Operators

Definition 5.1. A bicomplex linear operator U is called *unitary* if $U^*U = I$.

From Definition 5.1 one easily sees that the action of a bicomplex unitary operator preserves scalar products. Indeed let $|\psi\rangle, |\phi\rangle \in M$ and let U be unitary. Then

$$(U|\psi\rangle, U|\phi\rangle) = (U^*U|\psi\rangle, |\phi\rangle) = (I|\psi\rangle, |\phi\rangle) = (|\psi\rangle, |\phi\rangle) . \tag{5.1}$$

Lemma 5.2. *Let $U : M \to M$ be a unitary operator. Then $P_k(U) : V \to V$ $(k = 1, 2)$ is a unitary operator on V.*

Proof. From (4.13) and Theorem 3.11 we can write

$$P_k(U)^* P_k(U) = P_k(U^*)P_k(U) = P_k(U^*U) = P_k(I) = I. \qquad \square$$

We note that a bicomplex unitary operator cannot be in the null cone. For if it were, its determinant would also be in the null cone and the operator would not have an inverse.

Theorem 5.3. *Any eigenvalue λ of a bicomplex unitary operator satisfies $\lambda^{\dagger_3}\lambda = 1$.*

Proof. Let $|\phi\rangle \in M$ be an eigenket of a unitary operator U, associated with the eigenvalue λ, so that $U|\phi\rangle = \lambda|\phi\rangle$. Since U preserves scalar products, we can write

$$(|\phi\rangle, |\phi\rangle) = (U|\phi\rangle, U|\phi\rangle) = (\lambda|\phi\rangle, \lambda|\phi\rangle) = \lambda^{\dagger_3}\lambda\,(|\phi\rangle, |\phi\rangle) .$$

Since an eigenket is not in the null cone, $\lambda^{\dagger_3}\lambda = 1$ or, equivalently, $\lambda^{\dagger_3} = \lambda^{-1}$. $\square$

Corollary 5.4. *Let U be a unitary operator and let $|\phi\rangle \in M$ be an eigenket of U associated with the eigenvalue λ. Then $U^*|\phi\rangle = \lambda^{\dagger_3}|\phi\rangle$.*

Proof. Because U is unitary, one can write

$$\lambda^{\dagger_3}|\phi\rangle = \lambda^{\dagger_3}I|\phi\rangle = \lambda^{\dagger_3}U^*U|\phi\rangle = \lambda^{\dagger_3}\lambda U^*|\phi\rangle.$$

The result follows from Theorem 5.3. $\square$

Theorem 5.5. *Two eigenkets of a bicomplex unitary operator are orthogonal if the difference of the eigenvalues is not in $\mathcal{NC}$.*

Proof. Let $U : M \to M$ be a unitary operator and $|\phi\rangle, |\phi'\rangle$ be two eigenkets of U associated with eigenvalues λ and λ', respectively. Corollary 5.4 then implies

$$\begin{aligned} 0 &= (|\phi\rangle, U|\phi'\rangle) - (|\phi'\rangle, U^*|\phi\rangle)^{\dagger_3} = \lambda'\,(|\phi\rangle, |\phi'\rangle) - \left[\lambda^{\dagger_3}\,(|\phi'\rangle, |\phi\rangle)\right]^{\dagger_3} \\ &= \lambda'\,(|\phi\rangle, |\phi'\rangle) - \lambda\,(|\phi\rangle, |\phi'\rangle) = (\lambda' - \lambda)\,(|\phi\rangle, |\phi'\rangle) . \end{aligned}$$

If $\lambda' - \lambda \notin \mathcal{NC}$, then $(|\phi\rangle, |\phi'\rangle) = 0$. $\square$

5.2. Functions of an Operator

Let M be a finite-dimensional free $\mathbb{T}$-module and let A be a linear operator acting on M. Let $A^0 := I$ and let $\{c_n \mid n = 0, 1, \dots\}$ be an infinite sequence of bicomplex numbers. Formally we can write the infinite sum

$$\sum_{n=0}^{\infty} c_n A^n. \tag{5.2}$$

When this series converges to an operator acting on M, we call this operator $f(A)$.

The operator A and the coefficients c_n can be written in the idempotent basis as

$$A = \mathbf{e_1} A_{\hat{1}} + \mathbf{e_2} A_{\hat{2}}, \qquad c_n = \mathbf{e_1} c_{n\hat{1}} + \mathbf{e_2} c_{n\hat{2}}. \tag{5.3}$$

Substituting (5.3) into (5.2), we get

$$\begin{aligned} f(A) = \sum_{n=0}^{\infty} c_n A^n &= \mathbf{e_1} \sum_{n=0}^{\infty} c_{n\hat{1}} A_{\hat{1}}^n + \mathbf{e_2} \sum_{n=0}^{\infty} c_{n\hat{2}} A_{\hat{2}}^n \\ &= \mathbf{e_1} f_1(A_{\hat{1}}) + \mathbf{e_2} f_2(A_{\hat{2}}). \end{aligned} \tag{5.4}$$

One can see that the f series converges if and only if the two series f_1 and f_2 converge. These two are power series of operators acting in a finite-dimensional complex vector space.

A very important bicomplex function of an operator is of course the *exponential*, defined in the usual way as

$$\exp\{A\} = I + \sum_{n=1}^{\infty} \frac{1}{n!} A^n. \tag{5.5}$$

Clearly,

$$\exp\{A\} = \mathbf{e_1} \exp\{A_{\hat{1}}\} + \mathbf{e_2} \exp\{A_{\hat{2}}\}. \tag{5.6}$$

We now prove two important theorems on exponentials of operators.

Theorem 5.6. *If t is a real parameter, $\frac{d}{dt} exp\{tA\} = A exp\{tA\}$.*

Proof.

$$\begin{aligned} \frac{d}{dt}\exp\{tA\} &= \frac{d}{dt}\left[\mathbf{e_1} \exp\{tA_{\hat{1}}\} + \mathbf{e_2} \exp\{tA_{\hat{2}}\}\right] \\ &= \mathbf{e_1} A_{\hat{1}} \exp\{tA_{\hat{1}}\} + \mathbf{e_2} A_{\hat{2}} \exp\{tA_{\hat{2}}\} = A \exp\{tA\}. \end{aligned}$$ □

Theorem 5.7. *If H is self-adjoint, $exp\{\mathbf{i_1} H\}$ is unitary.*

Proof. Since $H_{\hat{1}}$ and $H_{\hat{2}}$ are self-adjoint in the usual (complex) sense, we have

$$\begin{aligned}
&[\exp\{\mathbf{i_1}H\}]^* \exp\{\mathbf{i_1}H\} \\
&= \left[\mathbf{e_1}\exp\left\{\mathbf{i_1}H_{\hat{1}}\right\} + \mathbf{e_2}\exp\left\{\mathbf{i_1}H_{\hat{2}}\right\}\right]^* \left[\mathbf{e_1}\exp\left\{\mathbf{i_1}H_{\hat{1}}\right\} + \mathbf{e_2}\exp\left\{\mathbf{i_1}H_{\hat{2}}\right\}\right] \\
&= \left[\mathbf{e_1}\exp\left\{-\mathbf{i_1}H_{\hat{1}}\right\} + \mathbf{e_2}\exp\left\{-\mathbf{i_1}H_{\hat{2}}\right\}\right] \left[\mathbf{e_1}\exp\left\{\mathbf{i_1}H_{\hat{1}}\right\} + \mathbf{e_2}\exp\left\{\mathbf{i_1}H_{\hat{2}}\right\}\right] \\
&= \mathbf{e_1}\exp\left\{-\mathbf{i_1}H_{\hat{1}}\right\}\exp\left\{\mathbf{i_1}H_{\hat{1}}\right\} + \mathbf{e_2}\exp\left\{-\mathbf{i_1}H_{\hat{2}}\right\}\exp\left\{\mathbf{i_1}H_{\hat{2}}\right\} \\
&= \mathbf{e_1}I + \mathbf{e_2}I = I.
\end{aligned}$$

□

5.3. Evolution Operator

A generalization of the Schrödinger equation to bicomplex numbers was proposed in [3]. It can be adapted to finite-dimensional modules as

$$\mathbf{i_1}\hbar\frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle, \tag{5.7}$$

where H is a self-adjoint bicomplex operator (called the Hamiltonian). Note that there is no gain in generality if one adds an arbitrary invertible bicomplex constant ξ on the left-hand side, i.e.

$$\mathbf{i_1}\xi\hbar\frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle. \tag{5.8}$$

Indeed one can then write

$$\mathbf{i_1}\hbar\frac{d}{dt}|\psi(t)\rangle = H'|\psi(t)\rangle, \tag{5.9}$$

with $H' = \xi^{-1}H$. For H' to be self-adjoint one must have $\xi^{\dagger_3} = \xi$, so that $\xi = \mathbf{e_1}\xi_{\hat{1}} + \mathbf{e_2}\xi_{\hat{2}}$, with $\xi_{\hat{1}}$ and $\xi_{\hat{2}}$ real. In this case (5.8) amounts to (5.7) with a redefinition of the Hamiltonian.

From Theorems 5.6 and 5.7 we immediately obtain

Theorem 5.8. *If H doesn't depend on time, solutions of* (5.7) *are given by $|\psi(t)\rangle = U(t,t_0)|\psi(t_0)\rangle$, where $|\psi(t_0)\rangle$ is any ket and*

$$U(t,t_0) = exp\left\{-\frac{\mathbf{i_1}}{\hbar}(t-t_0)H\right\}.$$

The operator $U(t,t_0)$ is unitary and is a generalization of the *evolution operator* of standard quantum mechanics [12].

6. Conclusion

We have derived a number of new results on finite-dimensional bicomplex matrices, modules, operators and Hilbert spaces, including the generalization of the spectral decomposition theorem. All these concepts are deeply connected with the formalism of quantum mechanics. We believe that many if not all of them can be extended to infinite-dimensional Hilbert spaces.

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Chapitre 4

Conclusion

How can the world be for quantum mechanics to be true ?
Louis Marchildon [53].

Dans ce mémoire, nous avons trouvé explicitement les valeurs propres ainsi que les fonctions propres de l'oscillateur harmonique quantique bicomplexe. Nous avons également développé un certain nombre d'outils mathématiques qui, nous croyons, constitueront la base de l'algèbre linéaire bicomplexe.

Bien que nous ayons trouvé de nouvelles valeurs propres et fonctions propres pour le problème de l'oscillateur harmonique, nous avons préféré laisser de côté, du moins pour l'instant, l'interprétation physique de ces résultats.

À l'aide des outils et des techniques utilisés dans ce mémoire, il semble raisonnable de penser que la plupart des problèmes de la mécanique quantique standard pourraient également être résolus à l'aide de la mécanique quantique bicomplexe. Ce dernier point semble donc ouvrir la porte à une large gamme d'applications de la mécanique quantique bicomplexe à toutes sortes de systèmes, incluant des systèmes plus « physiques » comme par exemple l'atome d'hydrogène ou les ions hydrogénoïdes.

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