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A GENERALIZED MANDELBROT SET FOR BICOMPLEX NUMBERS*

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Abstract

We use a commutative generalization of complex numbers called bicomplex numbers to introduce bicomplex dynamics. In particular, we give a generalization of the Mandelbrot set and of the "filled-Julia" sets in dimensions three and four. Also, we establish that our version of the Mandelbrot set with quadratic polynomial in bicomplex numbers of the form $w^2 + c$ is identically the set of points where the associated generalized "filled-Julia" set is connected. Moreover, we prove that our generalized Mandelbrot set of dimension four is connected.

1. INTRODUCTION

In 1982, A. Norton (1982)[1] gave some straightforward algorithms for the generation and display in 3-D of fractal shapes. For the first time, iteration with quaternions[2] appeared. Subsequently, theoretical results have been treated in Ref. 3 (1995) for the quaternionic Mandelbrot set defined with quadratic polynomial in the quaternions of the form q^2+c. However, in Ref. 4, S. Bedding and K. Briggs (1995) established that there is no interesting dynamics for this approach and it does not play any fundamental role analogous to that for the map $z^2 + c$ in the complex plane. We note that another definition of a Mandelbrot set for the quaternions was introduced by J. Holbrook (1987)[5]. This definition gives a Mandelbrot set in $\mathbb{R}^3$ which is not a slice of the quaternionic quadratic Mandelbrot set.

In this article, we use a commutative generalization of the complex numbers called bicomplex numbers[6–9] to give a new version of the Mandelbrot set in dimensions three and four. Moreover, we prove that our generalization in dimension four, noted $\mathcal{M}_2$, is a connected set. We also define the

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concept of "filled-Julia" set for bicomplex numbers and we prove that a point is inside $\mathcal{M}_2$ if and only if the "filled-Julia" set at that point is connected. These two results are perfectly analogous to the corresponding results in the complex plane.

Our generalization of the Mandelbrot set in dimension three is established from a slice of $\mathcal{M}_2$. We also give a graphics representation of our set, called the Tetrabrot, in $\mathbb{R}^3$ and we especially focus our attention on the infinite divergence layers to approach this set. Moreover, we give a graphics representation of the associated "filled-Julia" set for points on the Tetrabrot, and we note that shapes of certain "filled-Julia" sets are reflected in the shape of the Tetrabrot near the corresponding points. This feature had also been remarked for the Mandelbrot set in the complex plane.

Finally, we remark that the Tetrabrot could possibly be unconnected, and we establish hypotheses about the geometry of the Mandelbrot set for which the Tetrabrot would be unconnected.

2. PRELIMINARIES

Here, we introduce some of the basic results of the theory of bicomplex numbers. First, we define bicomplex numbers as follows: $\mathbb{C}_2 := \{a + bi_1 + ci_2 + dj \ : \ {i_1}^2 = {i_2}^2 = -1, j^2 = 1$ and $i_2 j = j i_2 = -i_1, i_1 j = j i_1 = -i_2, i_2 i_1 = i_1 i_2 = j\}$ where $a, b, c, d \in \mathbb{R}$. In this article, the norm used on $\mathbb{C}_2$ will be the Euclidean norm (also noted $|\ |$) of $\mathbb{R}^4$.

We remark that we can write a bicomplex number $a+bi_1+ci_2+dj$ as $(a+bi_1)+(c+di_1)i_2 = z_1+z_2i_2$ where $z_1, z_2 \in \mathbb{C}_1 := \{x+yi_1 : {i_1}^2 = -1\}$. Thus, $\mathbb{C}_2$ can be viewed as the complexification of the usual complex numbers $\mathbb{C}_1$ and a bicomplex number can be seen as an element of $\mathbb{C}^2$. Moreover, the norm of the bicomplex number is the same as the norm of the associated element (z_1, z_2) of $\mathbb{C}^2$. It is easy to see[6,8] that $\mathbb{C}_2$ is a commutative unitary ring with the following characterization for the noninvertible elements:

Proposition 1. *Let $w = a + bi_1 + ci_2 + dj \in \mathbb{C}_2$. Then w is noninvertible iff ($a = -d$ and $b = c$) or ($a = d$ and $b = -c$) iff ${z_1}^2 + {z_2}^2 = 0$.*

It is also important to know that every bicomplex number $z_1 + z_2 i_2$ has the following unique idempotent representation:

$$z_1 + z_2 i_2 = (z_1 - z_2 i_1)e_1 + (z_1 + z_2 i_1)e_2$$

where $e_1 = \frac{1+j}{2}$ and $e_2 = \frac{1-j}{2}$.

This representation is very useful because: addition, multiplication and division can be done term-by-term. Also, an element will be noninvertible iff $z_1 - z_2 i_1 = 0$ or $z_1 + z_2 i_1 = 0$. The next definition will be useful to construct a natural "disc" in $\mathbb{C}_2$.

Definition 1. We say that $X \subseteq \mathbb{C}_2$ is a $\mathbb{C}_2$-cartesian set determined by X_1 and X_2 if $X = X_1 \times_e X_2 := \{z_1 + z_2 i_2 \in \mathbb{C}_2 \ : \ z_1 + z_2 i_2 = w_1 e_1 + w_2 e_2, (w_1, w_2) \in X_1 \times X_2\}$.

In Ref. 6, it is shown that if X_1 and X_2 are domains of $\mathbb{C}_1$ then $X_1 \times_e X_2$ is also a domain of $\mathbb{C}_2$. Then, a manner to construct a natural "disc" in $\mathbb{C}_2$ is to take the $\mathbb{C}_2$-cartesian product of two discs in $\mathbb{C}_1$. Hence, we define the natural "disc" of $\mathbb{C}_2$ as follows[6]: $D(0, r) := B^1(0, r) \times_e B^1(0, r) = \{z_1 + z_2 i_2 \ : \ z_1 + z_2 i_2 = w_1 e_1 + w_2 e_2, |w_1| < r, |w_2| < r\}$ where $B^n(0, r)$ is the open ball of $\mathbb{C}_1^n \simeq \mathbb{C}^n$ with radius r.

3. THE GENERALIZED MANDELBROT SET

In this section, we want to give a version of the Mandelbrot set for the bicomplex numbers. First, we recall the definition of the Mandelbrot set for the complex plane:

Definition 2. Let $P_c(z) = z^2 + c$ where $z, c \in \mathbb{C}$ and $P_c^{\circ n} := (P_c^{\circ(n-1)} \circ P_c)(z)$. Then the Mandelbrot set is defined as follows: $\mathcal{M} = \{c \in \mathbb{C} : P_c^{\circ n}(0)$ is bounded $\forall n \in \mathbb{N}\}$. When we take $z, c \in \mathbb{C}_1$, we denote the Mandelbrot set by $\mathcal{M}_1$.

Figure 1 gives an illustration of the Mandelbrot set with some of its "filled-Julia" sets. In fact, our figure is a rotation by 90° of the original Mandelbrot set. This rotation will give a better vantage point when we shall work on our version of the Mandelbrot set in $\mathbb{R}^3$. Also, the colors around the Mandelbrot set have been determined by the number of iterations needed before $|P_c^{\circ n}(0)| > 2$. This is well justified by the fact that the Mandelbrot set can also be characterized as follows: $\mathcal{M} = \{c \in \mathbb{C} : |P_c^{\circ n}(0)| \leq 2 \ \forall n \in \mathbb{N}\}$.[11] Then, the colors give information about the manner in which

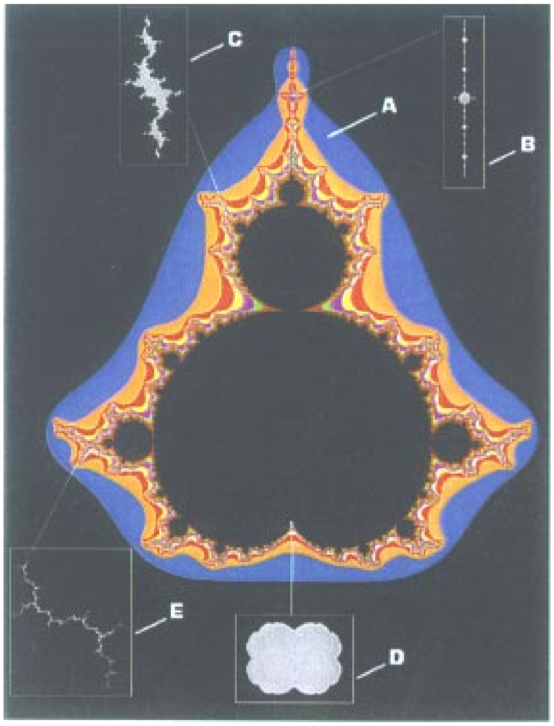

Fig. 1

the algorithm for the Mandelbrot set diverges to infinity. This information will be almost the only possible one to approach our version of the Mandelbrot set in dimension three.

We also recall the following beautiful property of the Mandelbrot set:[10]

Theorem 1.[10] *The Mandelbrot set $\mathcal{M}$ is connected.*

Now, to give a version of the Mandelbrot set for the bicomplex numbers we have only to reproduce the algorithm of Definition 2 for the bicomplex numbers. This is the next definition.

Definition 3. Let $P_c(w) = w^2 + c$ where $w, c \in \mathbb{C}_2$ and $P_c^{\circ n}(w) := (P_c^{\circ(n-1)} \circ P_c)(w)$. Then the generalized Mandelbrot set for bicomplex numbers is defined as follows: $\mathcal{M}_2 = \{c \in \mathbb{C}_2 : P_c^{\circ n}(0)$ is bounded $\forall n \in \mathbb{N}\}$.

The next lemma is a characterization of $\mathcal{M}_2$ using only $\mathcal{M}_1$. This lemma will be useful to prove that $\mathcal{M}_2$ is also a connected set.

Lemma 1. $\mathcal{M}_2 = \mathcal{M}_1 \times_e \mathcal{M}_1$.

Proof. First, we prove that $\mathcal{M}_2 \subseteq \mathcal{M}_1 \times_e \mathcal{M}_1$. Let $c \in \mathbb{C}_2$ such that $P_c^{\circ n}(0)$ is bounded $\forall n \in \mathbb{N}$. We have

$$\begin{aligned} P_c(w) &= w^2 + c \\ &= [(z_1 - z_2 i_1)^2 + (c_1 - c_2 i_1)]e_1 \\ &\quad + [(z_1 + z_2 i_1)^2 + (c_1 + c_2 i_1)]e_2 \end{aligned}$$

where $w = (z_1 - z_2 i_1)e_1 + (z_1 + z_2 i_1)e_2$ and $c = (c_1 - c_2 i_1)e_1 + (c_1 + c_2 i_1)e_2$. Then,

$$P_c^{\circ n}(w) = P_{c_1 - c_2 i_1}^{\circ n}(z_1 - z_2 i_1)e_1 + P_{c_1 + c_2 i_1}^{\circ n}(z_1 + z_2 i_1)e_2 .$$

By hypothesis,

$$\begin{aligned} P_c^{\circ n}(0) &= P_{c_1 - c_2 i_1}^{\circ n}(0)e_1 \\ &\quad + P_{c_1 + c_2 i_1}^{\circ n}(0)e_2 \text{ is bounded } \forall n \in \mathbb{N}. \end{aligned}$$

Hence, $P_{c_1 - c_2 i_1}^{\circ n}(0)$ and $P_{c_1 + c_2 i_1}^{\circ n}(0)$ are also bounded $\forall n \in \mathbb{N}$. Then $c_1 - c_2 i_1$, $c_1 + c_2 i_1 \in \mathcal{M}_1$ and $c = (c_1 - c_2 i_1)e_1 + (c_1 + c_2 i_1)e_2 \in \mathcal{M}_1 \times_e \mathcal{M}_1$.

Conversely, if we take $c \in \mathcal{M}_1 \times_e \mathcal{M}_1$, we have $c = (c_1 - c_2 i_1)e_1 + (c_1 + c_2 i_1)e_2$ with $c_1 - c_2 i_1, c_1 + c_2 i_1 \in \mathcal{M}_1$. Hence, $P_{c_1 - c_2 i_1}^{\circ n}(0)$ and $P_{c_1 + c_2 i_1}^{\circ n}(0)$ are also bounded $\forall n \in \mathbb{N}$. Then $P_c^{\circ n}(0)$ is bounded $\forall n \in \mathbb{N}$, that is $c \in \mathcal{M}_2$. □

Theorem 2. *The generalized Mandelbrot set $\mathcal{M}_2$ is connected.*

Proof. Define a mapping e as follows:

$$\mathbb{C}_1^2 = \mathbb{C}_1 \times \mathbb{C}_1 \xrightarrow{e} \mathbb{C}_1 \times_e \mathbb{C}_1 = \mathbb{C}_2$$

$$(w_1, w_2) \mapsto w_1 e_1 + w_2 e_2 .$$

The mapping e is clearly a homeomorphism. Then, if X_1 and X_2 are connected subsets of $\mathbb{C}_1$ we have that $e(X_1 \times X_2) = X_1 \times_e X_2$ is also connected. Now, by Lemma 1, $\mathcal{M}_2 = \mathcal{M}_1 \times_e \mathcal{M}_1$. Moreover, by Theorem 1, $\mathcal{M}_1$ is connected. It follows, if we let $X_1 = X_2 = \mathcal{M}_1$, that $\mathcal{M}_2$ is connected. □

4. THE TETRABROT

In the previous section, we established a version of the Mandelbrot set in dimension four. Now, we want to give a version of the Mandelbrot set in dimension three using the definition for $\mathcal{M}_2$. The idea here is to preserve the Mandelbrot set inside $\mathcal{M}_2$. Then, if we restrict the algorithm to the points of the form $a + bi_1 + ci_2$ where $a, b, c \in \mathbb{R}$, we preserve the Mandelbrot set on two perpendicular complex planes and we stay in $\mathbb{R}^3$. This is the first argument to justify the following definition.

Definition 4. The "Tetrabrot" is defined as follows: $\mathcal{T} = \{c = c_1 + c_2 i_2 \in \mathbb{C}_2 : \text{Im}(c_2) = 0$ and $P_c^{\circ n}(0)$ is bounded $\forall n \in \mathbb{N}\}$.

We wish to give an illustration of the Tetrabrot in $\mathbb{R}^3$. The next result will give a manner to approach the Tetrabrot with the Euclidean norm in $\mathbb{R}^4$.

Theorem 3. $\mathcal{M}_2 \subset \bar{\mathbf{D}}(0, 2) \subset \overline{\mathbf{B}^2(0, 2)}$ *where* $\bar{\mathbf{D}}(0, 2) = \overline{\mathbf{B^1(0, 2)}} \times_e \overline{\mathbf{B^1(0, 2)}}$. *Moreover, the radius 2 is the best possible in each case.*

Proof. By Lemma 1, $\mathcal{M}_2 = \mathcal{M}_1 \times_e \mathcal{M}_1$. Moreover, $\bar{\mathbf{D}}(0, 2) = \overline{\mathbf{B^1(0, 2)}} \times_e \overline{\mathbf{B^1(0, 2)}}$ and $\mathcal{M}_1 \subset \overline{\mathbf{B^1(0, 2)}}$ with a point of $\mathcal{M}_1$ which touches the boundary of this disc.[11] Then, $\mathcal{M}_2 \subset \bar{\mathbf{D}}(0, 2)$ and the radius 2 is the best possible. Finally, it is proven in Ref. 6 that $\bar{\mathbf{D}}(0, 2) \subset \overline{\mathbf{B^2(0, 2)}}$ with points of $\bar{\mathbf{D}}(0, 2)$ which touch the boundary of $\overline{\mathbf{B^2(0, 2)}}$. $\square$

Then, it is possible to compute the infinite divergence layers of the Tetrabrot from the number of iterations needed to have $|P_c^{\circ n}(0)| > 2$. We have to remark here that each divergence layer will hide the others. For example, Fig. 2 is an illustration for the Tetrabrot of one of its divergence layers in correspondence with the divergence layer illustrated in Fig. 1(A) for the Mandelbrot set. In fact, the Tetrabrot is inside Fig. 2. It is possible to see a part of the Tetrabrot (see Fig. 3) if we cut a piece of Fig. 2. In Fig. 3, the colors are an illustration of the other divergence layers. It is also possible to compute other divergence layers (see Figs. 4–7). Figure 7 begins to be close to the set we wish to approach; then Fig. 7 with its cut plane gives certainly a good idea of the Tetrabrot.

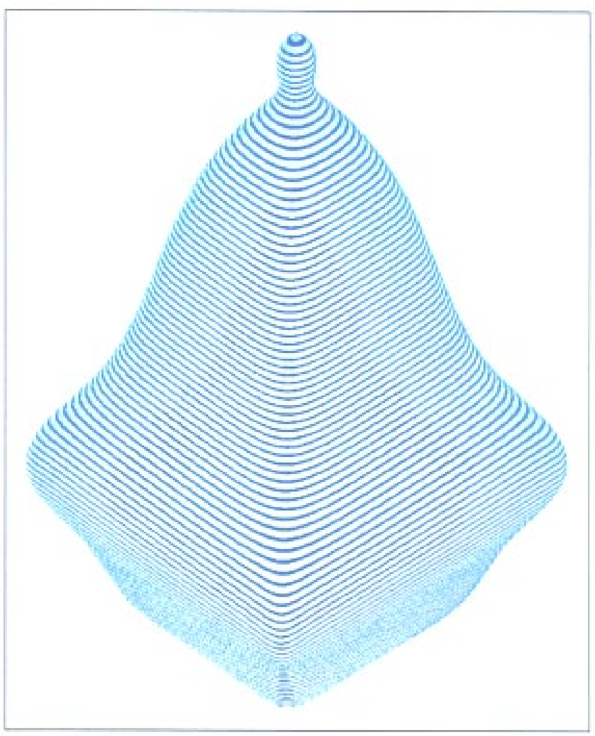

Fig. 2

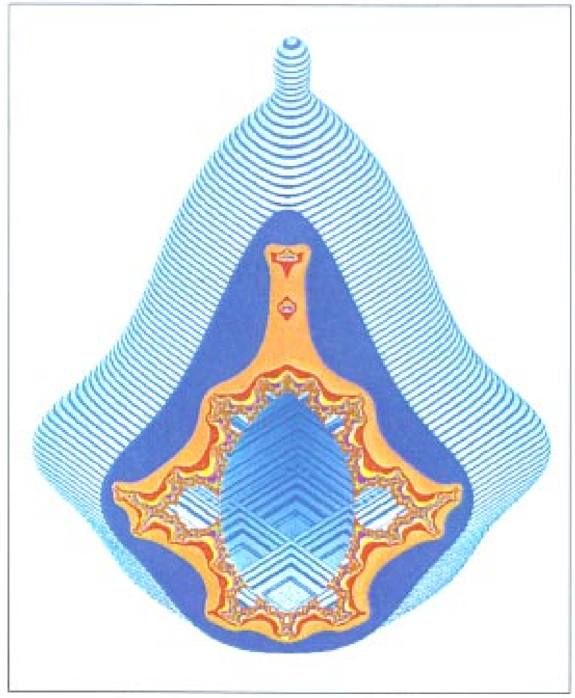

Fig. 3

To define the Tetrabrot, we have put the last coordinate in "j" equal to zero. In fact it is possible to do the same thing if we fix the last coordinate equal to a number different from zero. However, if we do that, we lose the beautiful symmetry of the Tetrabrot. Figures 8 and 9 give an illustration of this phenomenon for two different fixed "dj" with $d \neq 0$.

An interesting exploration of the Tetrabrot is now possible. For example, Fig. 10 is an enlargement of Fig. 7(A). It is also possible to be more specific. For example, Fig. 14 is an enlargement of 10(A) and Figs. 11 and 12 are an enlargement of deep zones above the zone of Fig 7(A). Also, Fig. 13

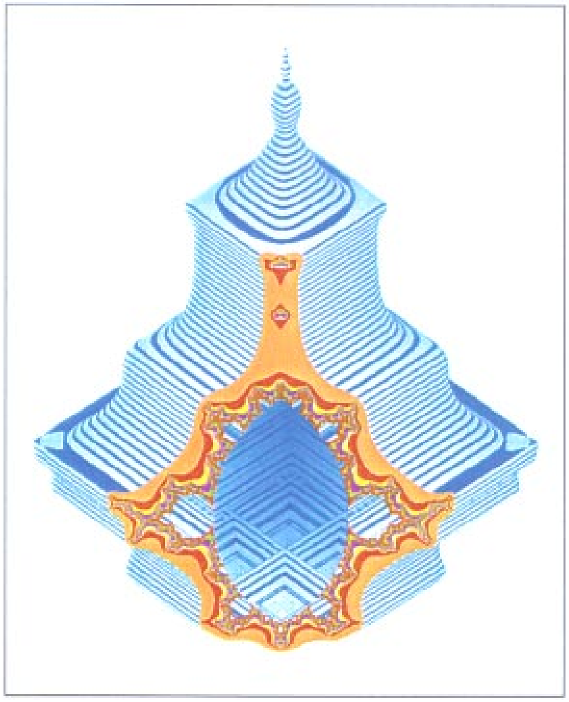

Fig. 4

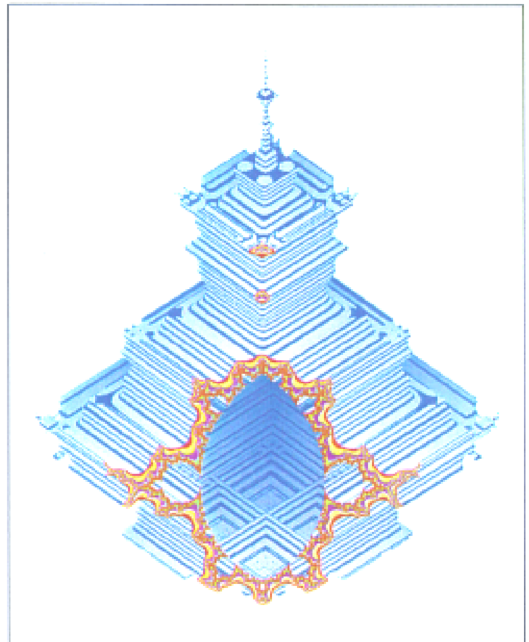

Fig. 6

Fig. 5

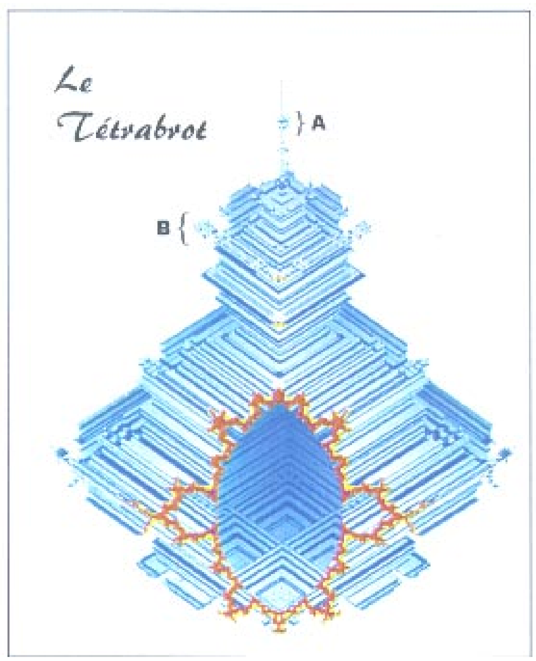

Fig. 7

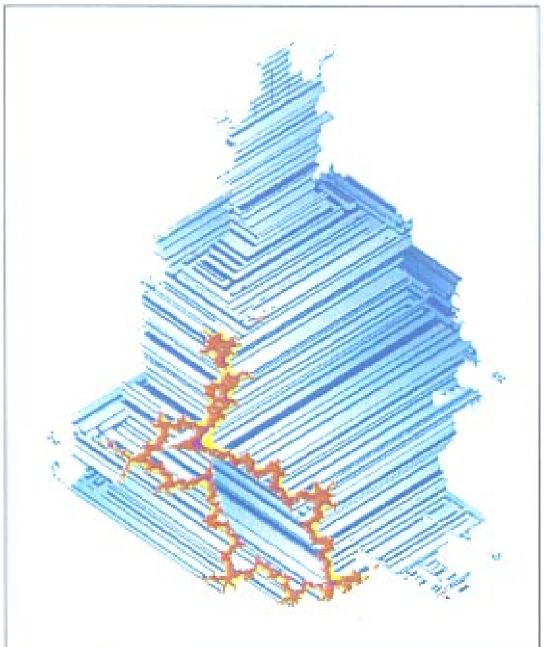

Fig. 8

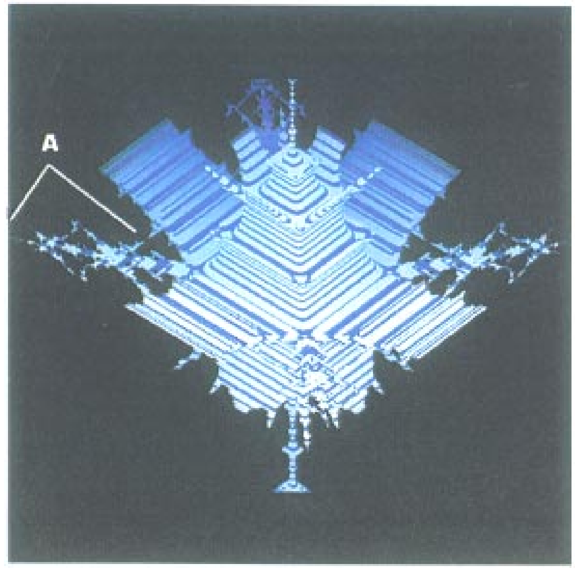

Fig. 10

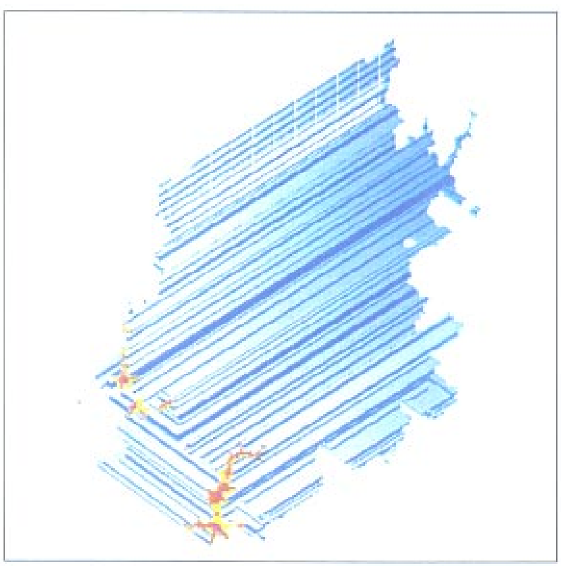

Fig. 9

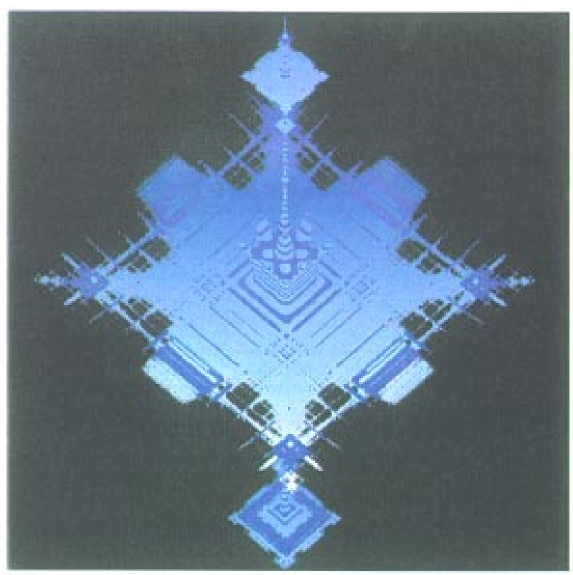

Fig. 11

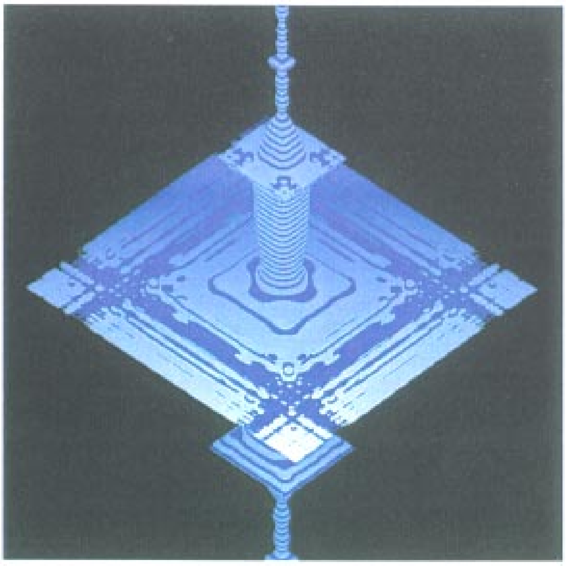

Fig. 12

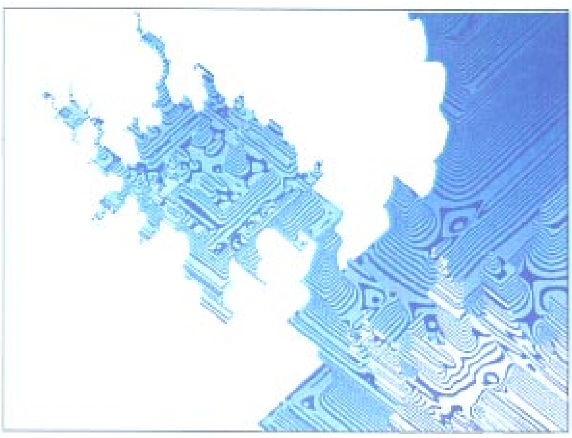

Fig. 13

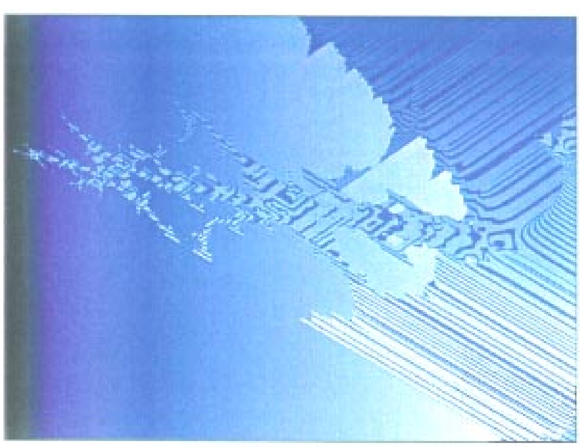

Fig. 14

is an enlargement of Fig. 7(B). The color in the background of Fig. 14 has been added to give a better 3-D view. Each figure has been illustrated with a selected divergence layer and striations have been added to give an illustration of the "level curves" of each figure.

5. THE GENERALIZED "FILLED-JULIA" SET

It is now interesting to see what happens with the Julia set. First, we recall the definition of the "filled-Julia" set in the complex plane:

Definition 5. The "filled-Julia" set is defined as follows: $(c \in \mathbb{C})$

$$\mathcal{K}_c = \{z \in \mathbb{C} : P_c^{\circ n}(z) \text{ is bounded } \forall n \in \mathbb{N}\}\,.$$

Moreover, the Julia set $\mathcal{J}_c := \partial\mathcal{K}_c$.

We recall also the following beautiful relationship between the Mandelbrot set and the "filled-Julia" set:

Theorem 4. *$c \in \mathcal{M} \Leftrightarrow \mathcal{K}_c$ is connected.*

It is possible to generalize the "filled-Julia" set for bicomplex numbers:

Definition 6. The generalized "filled-Julia" set for bicomplex numbers is defined as follows: $(c \in \mathbb{C}_2)$

$$\mathcal{K}_{2,c} = \{w \in \mathbb{C}_2 : P_c^{\circ n}(w) \text{ is bounded } \forall n \in \mathbb{N}\}\,.$$

The next lemma gives a characterization of the "filled-Julia" set for bicomplex numbers in terms of the "filled-Julia" set for the complex plane. This lemma will be useful to give an analogue of Theorem 4 for the bicomplex numbers.

Lemma 2. $\mathcal{K}_{2,c} = \mathcal{K}_{2,(c_1-c_2i_1)e_1+(c_1+c_2i_1)e_2} = \mathcal{K}_{c_1-c_2i_1} \times_e \mathcal{K}_{c_1+c_2i_1}$.

Proof. The proof is along the same lines as the proof of the Lemma 1. □

Theorem 5. *$c \in \mathcal{M}_2 \Leftrightarrow \mathcal{K}_{2,c}$ is connected.*

Proof. By Lemma 2, we know that $\mathcal{K}_{2,c} = \mathcal{K}_{c_1-c_2i_1} \times_e \mathcal{K}_{c_1+c_2i_1}$. Also, by the homeomorphism in the proof of Theorem 2, $\mathcal{K}_{c_1-c_2i_1} \times_e \mathcal{K}_{c_1+c_2i_1}$ is connected if and only if $\mathcal{K}_{c_1-c_2i_1} \times \mathcal{K}_{c_1+c_2i_1}$ is connected. Then, $\mathcal{K}_{c_1-c_2i_1} \times_e \mathcal{K}_{c_1+c_2i_1}$ is connected if and only if $\mathcal{K}_{c_1-c_2i_1}$ and $\mathcal{K}_{c_1+c_2i_1}$ are connected. Hence, by Theorem 4, $\mathcal{K}_{2,c}$ is connected if and only if $c_1 - c_2i_1, c_1 + c_2i_1 \in \mathcal{M}_1$. Therefore, by Lemma 1, $\mathcal{K}_{2,c}$ is connected if and only if $c = (c_1 - c_2i_1)e_1 + (c_1 + c_2i_1)e_2 \in \mathcal{M}_2$. □

6. THE "FILLED-JULIA" SET FOR THE TETRABROT

The same process as for the Tetrabrot yields a version of the "filled-Julia" set in $\mathbb{R}^3$. We define the "filled-Julia" set for the Tetrabrot.

Definition 7. The associated "filled-Julia" set for the Tetrabrot is defined as follows: $(c \in \mathbb{C}_2)$

$$\mathcal{L}_{2,c} = \{w = w_1 + w_2i_2 \in \mathbb{C}_2 \, : \, \mathrm{Im}(w_2) = 0 \text{ and } P_c^{\circ n}(w) \text{ is bounded } \forall n \in \mathbb{N}\}.$$

Figure 15 is an illustration of the "filled-Julia" set for the Tetrabrot at the same point $c = -1.754878$ as the "filled-Julia" set B of Fig. 1. Hence, Fig. 15 is a kind of generalization of the "filled-Julia" set $\mathcal{K}_c$ in the complex plane. In the same manner, Fig. 16 is the generalization of Fig. 1(C), and Fig. 17 the generalization of Fig. 1(D).

In Ref. 11, L. Carleson and T. W. Gamelin (1993) have remarked this interesting fact: "One striking feature of $\mathcal{M}$ is that shapes of certain of the Julia

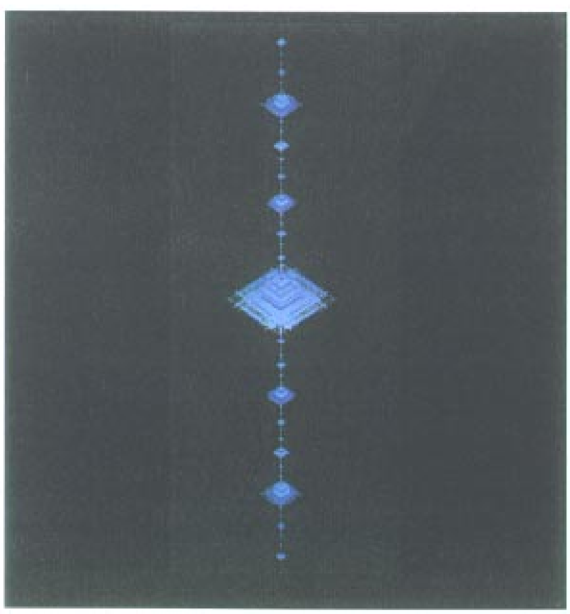

Fig. 15

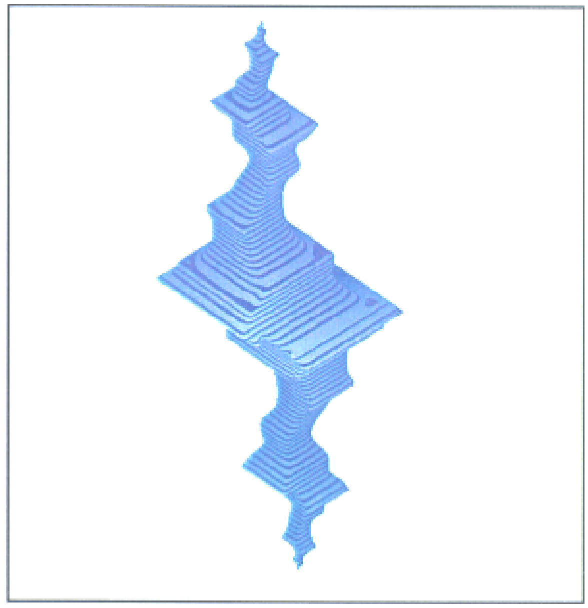

Fig. 16

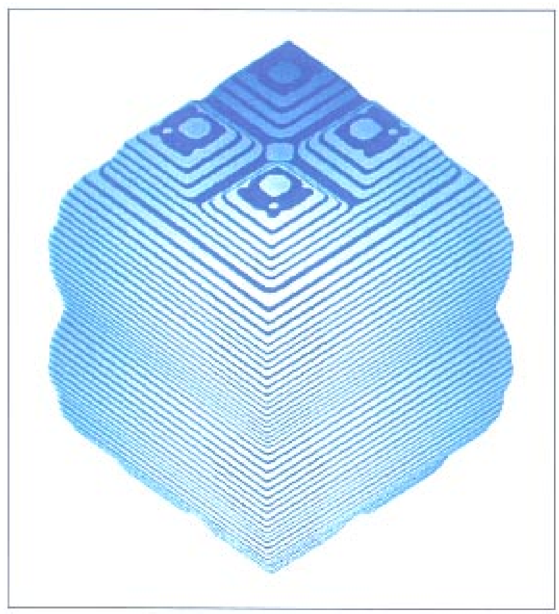

Fig. 17

sets $\mathcal{J}_c$ in dynamic space (z-space) are reflected in the shape (c-shape)". For the Tetrabrot, we seem to have something similar. For example, Fig. 18 is Fig. 17 with the same kind of cut as for the Tetrabrot in Fig. 7. Hence we see that inside Fig. 17, we have the same shape as inside the Tetrabrot near the point $c = 0.25$. It is also possible to see the same phenomenon with the "filled-Julia" set of

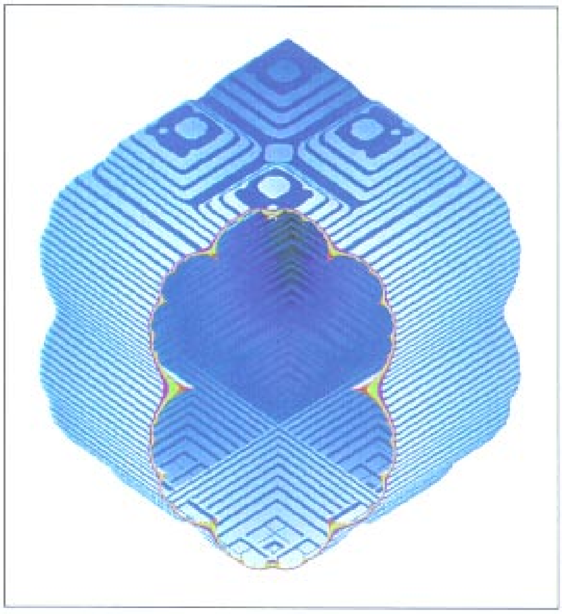

Fig. 18

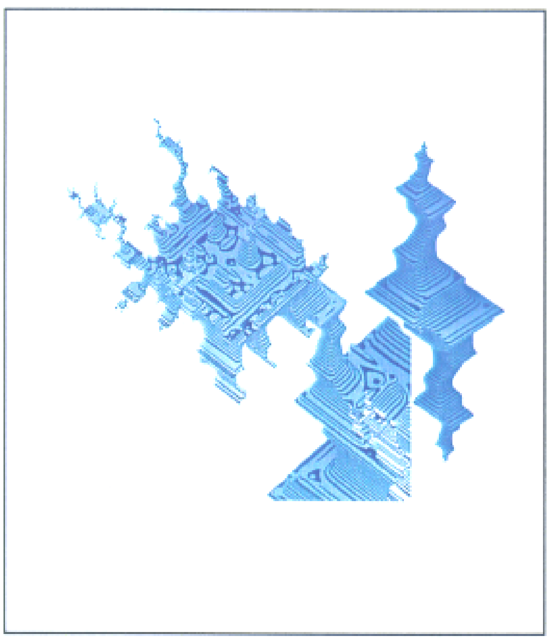

Fig. 19

Fig. 20

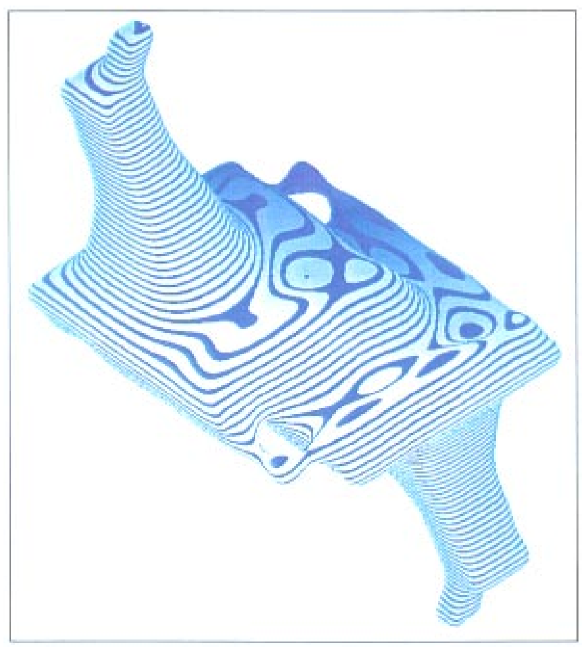

Fig. 21

Fig. 16. This phenomenon has been illustrated in Fig. 19 where we have put together the border of the Tetrabrot and the associated "filled-Julia" set at the point $c = -1.16 - 0.25i_1$ on the border. We see clearly that this "filled-Julia" set imitates the border of the Tetrabrot.

Finally, in Figs. 20–23, we show the "filled-Julia" set at $c = -i_1$ for different infinite divergence layers. We remark that Fig. 23 is a good approximation of this set and an interesting generation of Fig. 1-E.

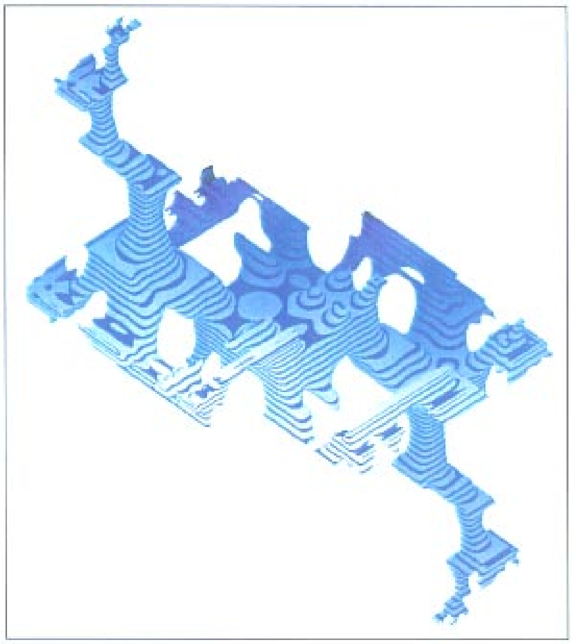

Fig. 22

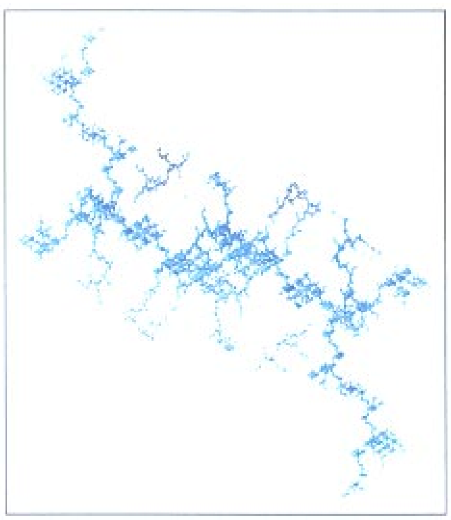

Fig. 23

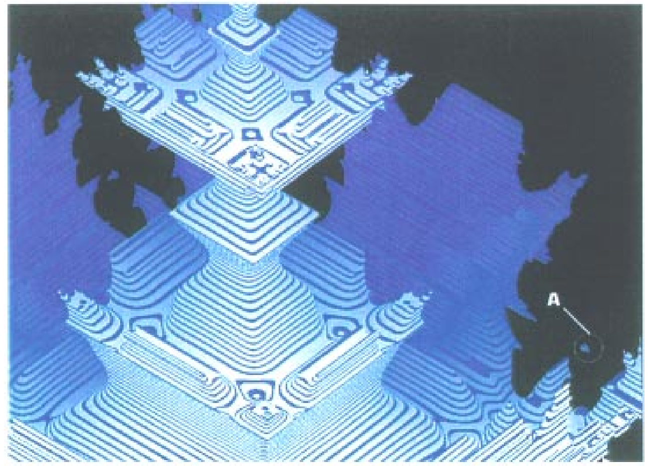

Fig. 24

7. CONJECTURE

We have proved in Sec. 3 that $\mathcal{M}_2$ is a connected set. It is natural to ask whether the Tetrabrot is also connected. Until now, the exploration of the Tetrabrot seems to confirm that the Tetrabrot is connected. However, if we enlarge Fig. 7 in the center of the Tetrabrot above the zone B, we notice [see Fig. 24(A)] that there is a piece which seems to be disconnected from the main figure (Fig. 25 focuses on this piece). Because we work with divergence layers and a computational approximation, we are far from knowing if the piece is really unconnected or if there is point inside the piece which is also inside the Tetrabrot. However this is enough to formulate a conjecture:

Conjecture 1. *The Tetrabrot is unconnected.*

It is possible to translate the conjecture into a question about the geometry of the Mandelbrot set. For this, we need to prove the following lemma which is itself of interest:

Lemma 3. *The Tetrabrot can be characterized as follows*:

$$\mathcal{T} = \bigcup_{y \in [-m,m]} \{[(\mathcal{M}_1 - yi_1) \cap (\mathcal{M}_1 + yi_1)] + yi_2\}$$

where $m := \sup\{q \in \mathbb{R} : \exists p \in \mathbb{R}$ *such that* $p + qi_1 \in \mathcal{M}_1\}$.

Proof. By definition,

$$\mathcal{T} = \{c = c_1 + c_2 i_2 \in \mathbb{C}_2 : \mathrm{Im}(c_2) = 0 \text{ and } P_c^{\circ n}(0) \text{ is bounded } \forall n \in \mathbb{N}\}\,.$$

Let $c = (c_1 - c_2 i_1)e_1 + (c_1 + c_2 i_1)e_2$ with $c_1 = c_{11} + c_{12}i_1$ and $c_2 = c_{21} + c_{22}i_1$ where c_{11}, c_{12}, c_{21}, $c_{22} \in \mathbb{R}$. Now, if $\mathrm{Im}(c_2) = 0$, we have $c_2 = c_{21} + 0i_1$ and therefore, $c = (c_1 - c_{21})e_1 + (c_1 + c_{21}i_1)e_2$ whenever $\mathrm{Im}(c_2) = 0$. Hence $\mathcal{T} = \{(c_1 - c_{21}i_1)e_1 + (c_1 + c_{21}i_1)e_2 : P_c^{\circ n}(0)$ is bounded

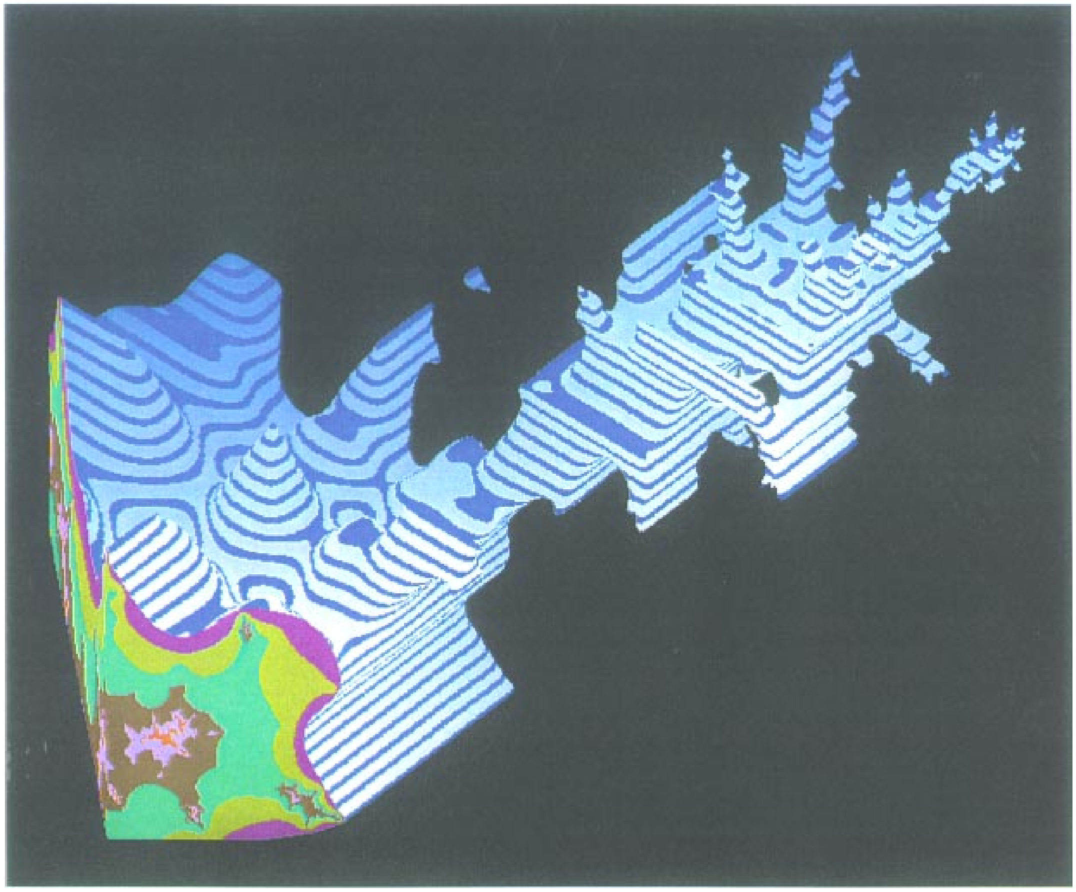

Fig. 25

$\forall n \in \mathbb{N}\} = \{(c_1 - c_{21}i_1)e_1 + (c_1 + c_{21}i_1)e_2 : P^{\circ n}_{c_1-c_{21}i_1}(0)$ and $P^{\circ n}_{c_1+c_{21}i_1}(0)$ are bounded $\forall n \in \mathbb{N}\}$. To continue the proof, we need to remark the following fact: $\forall z \in \mathbb{C}_1$,

$$\{c \in \mathbb{C}_1 : P^{\circ n}_{c+z}(0) \text{ is bounded } \forall n \in \mathbb{N}\} = \mathcal{M}_1 - z\,.$$

By definition, $P^{\circ n}_{c_1-c_{21}i_1}(0)$ and $P^{\circ n}_{c_1+c_{21}i_1}(0)$ are bounded $\forall n \in \mathbb{N}$ if and only if $c_1 - c_{21}i_1, c_1 + c_{21}i_1 \in \mathcal{M}_1$, and by the remark, it is also if and only if $c_1 \in (\mathcal{M}_1 - c_{21}i_1) \cap (\mathcal{M}_1 + c_{21}i_1)$. Hence, if we express $(c_1 - c_{21}i_1)e_1 + (c_1 + c_{21})e_2 = c_1 + c_{21}i_2 = c_{11} + c_{12}i_1 + c_{21}i_2$, the Tetrabrot can be characterized as follows:

$$\begin{aligned}\mathcal{T} &= \{c_{11} + c_{12}i_1 + c_{21}i_2 : c_{11} + c_{12}i_1 \\ &\quad \in (\mathcal{M}_1 - c_{21}i_1) \cap (\mathcal{M}_1 + c_{21}i_1)\} \\ &= \bigcup_{y\in\mathbb{R}} \{[(\mathcal{M}_1 - yi_1) \cap (\mathcal{M}_1 + yi_1)] + yi_2\}\,.\end{aligned}$$

It is possible to be more precise with the last expression. In fact,

$$= \bigcup_{y\in[-m,m]} \{[(\mathcal{M}_1 - yi_1) \cap (\mathcal{M}_1 + yi_1)] + yi_2\}$$

because $(\mathcal{M}_1 - yi_1) \cap (\mathcal{M}_1 + yi_1) = \emptyset$ whenever $y \in [-m, m]^c$. This comes from the fact that $\mathcal{M}_1 \subset \{z \in \mathbb{C}_1 : |\mathrm{Im}(z)| \le m\}$.

Moreover, $(\mathcal{M}_1 - yi_1) \cap (\mathcal{M}_1 + yi_1) \ne \emptyset\ \forall y \in [-m, m]$. To see this, we just have to prove that $E_y := \{c = c_{11} + 0i_1 + yi_2 : P^{\circ n}_c(0)$ is bounded $\forall n \in \mathbb{N}\}$ is nonempty $\forall y \in [-m, m]$ because $E_y \subset \{c = c_{11} + c_{12}i_1 + c_{21}i_2 : P^{\circ n}_c(0)$ is bounded $\forall n \in \mathbb{N}\} = \{c_{11} + c_{12}i_1 + c_{21}i_2 : c_{11} + c_{12}i_1 \in (\mathcal{M}_1 - c_{21}i_1) \cap (\mathcal{M}_1 + c_{21}i_1)\}$. In fact, the set E_y is the algorithm for the Mandelbrot set iterates, with the imaginary part in "i_2" fixed at y. By the compactness and the symmetry of the Mandelbrot set $\mathcal{M}_1$, there must exist x_m such that $x_m - mi_2$, $x_m + mi_2 \in E_m$. Therefore, because $\mathcal{M}_1$ is connected, we must have $E_y \ne \emptyset\ \forall y \in [-m, m]$.

Theorem 6. *Let*

$$R_1 := R(-1.3939 + 0.0848i_1;\ -1.3893 + 0.0803i_1)$$
$$L := R(-1.3939 + 0.0803i_1;\ -1.3939 + 0.0728i_1)$$
$$R_2 := R(-1.3939 + 0.0728i_1;\ -1.3893 + 0.0683i_1)$$
$$L_3 := R(-1.3939 + 0.1304i_1;\ -1.3893 + 0.1304i_1)$$
$$L_4 := R(-1.3939 + 0.1259i_1;\ -1.3893 + 0.1259i_1)$$
$$L_5 := R(-1.3893 + 0.1304i_1;\ -1.3893 + 0.1259i_1)$$
$$L_6 := R(-1.3939 + 0.1304i_1;\ -1.3939 + 0.1259i_1)$$

where

$$R(a + bi_1, c + di_1) := \left\{\alpha_1(a + bi_1) + \alpha_2(c + bi_1) + \alpha_3(a + di_1) + \alpha_4(c + di_1) \middle|\ \sum_{i=1}^{4} \alpha_i = 1, \alpha_i \geq 1\right\}.$$

If

$$F_1 := R_1 \cup L \cup R_2$$

and

$$F_2 := L_3 \cup L_4 \cup L_5$$

are disjoint from the Mandelbrot set, and

$$z_1^* := -1.391816306 + 0.129472959i_1$$

and

$$z_2^* := -1.392873019 + 0.077172405i_1$$

are inside the Mandelbrot set, then the Tetrabrot is unconnected.

Proof. The goal of the proof is to construct, from the hypothesis about the Mandelbrot set, a box where the algorithm for the Tetrabrot diverges with, inside the box, a point where the algorithm converges. The box that we want to construct is a box around the piece of Fig. 25. Also, to understand better with which zones we work, Fig. 26(A) gives an indication where the zones of the hypothesis are on the Mandelbrot set (Figs. 27–29 are enlargements of Fig. 26(A) where the specific sets F_1, F_2, z_1^* and z_2^* are illustrated).

The "box of divergence" is constructed as follows: let $y_1 := 0.0228$ and $y_2 := 0.0288$,

$$B_i := R_i + y_i i_1 - y_i i_2 \quad \text{for i} = 1,\ 2\,,$$

$$B_i := \bigcup_{y \in [y_1, y_2]} (L_i - y i_1 - y i_2) \quad \text{for i} = 3,\ \ldots,\ 6\,.$$

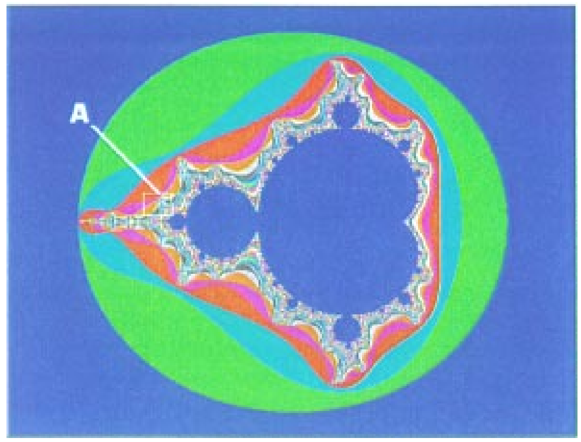

Fig. 26

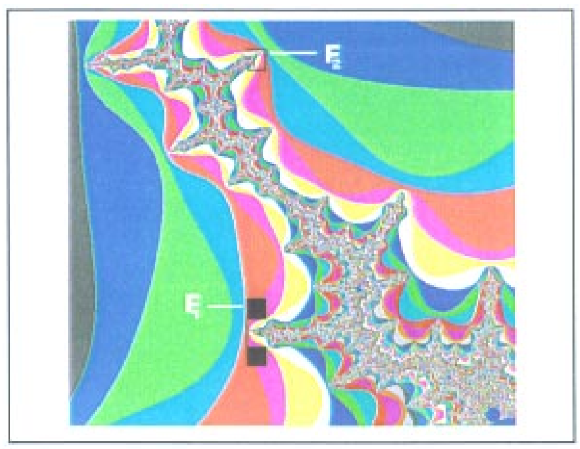

Fig. 27

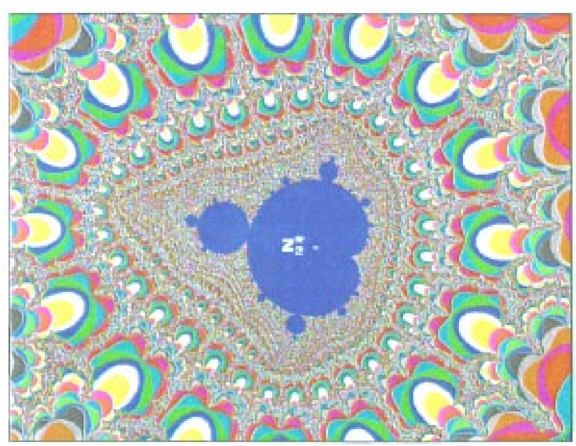

Fig. 28

Fig. 29

Each "B_i" is a side of the box; then the "box of divergence" is $B := \bigcup_{i=1}^{6} B_i$.

First, we have to confirm that each "B_i" is a set where the algorithm for the Tetrabrot diverges. This is possible with Lemma 3 and the assumption that $F_1 \cup F_2$ is not in the Mandelbrot set.

For B_1, by Lemma 3, we just need to prove that:

$$B_1 \cap ([(\mathcal{M}_1 - y_1 i_1) \cap (\mathcal{M}_1 + y_1 i_1)] - y_1 i_2) = \emptyset\,.$$

This is clear because if $B_1 \cap ([(\mathcal{M}_1 - y_1 i_1) \cap (\mathcal{M}_1 + y_1 i_1)] - y_1 i_2) \neq \emptyset$, we obtain that there exists $z_1 \in R_1$ such that $z_1 + y_1 i_1 \in [(\mathcal{M}_1 - y_1 i_1) \cap (\mathcal{M}_1 + y_1 i_1)]$. However, this is impossible because if $z_1 + y_1 i_1 \in \mathcal{M}_1 + y_1 i_1$ we obtain that $z_1 \in \mathcal{M}_1$ and this contradicts the hypothesis.

A similar proof is possible for B_2. The cases of B_3, B_4 and B_5 are also along the same lines. For example:

$$B_3 \cap ([(\mathcal{M}_1 - y i_1) \cap (\mathcal{M}_1 + y i_1)] - y i_2) = \emptyset \qquad \forall y \in [y_1,\, y_2]$$

because if it is not true, there must exist $y \in [y_1, y_2]$ and $z_3 \in L_3$ such that $z_3 - y i_1 \in [(\mathcal{M}_1 - y i_1) \cap (\mathcal{M}_1 + y i_1)]$. However, this is impossible because if $z_3 - y i_1 \in \mathcal{M}_1 - y i_1$, we obtain that $z_3 \in \mathcal{M}_1$ and this contradicts the hypothesis. For B_6, the same argument is possible if we remark that $L_6 - y i_1 \subset F_1 + y i_1$ $\forall y \in [y_1, y_2]$.

Now, we have to confirm that each B_i is on the same box. For this, we will just remark the following fact:

$$L_i - y_k i_1 - y_k i_2 \subset B_k = R_k + y_k i_1 - y_k i_2 \qquad \forall i = 3, \ldots, 6 \text{ and } \forall k = 1,\, 2$$

because $L_i - y_k i_1 \subset R_k + y_k i_1$ $\forall i = 3, \ldots, 6$ and $\forall k = 1, 2$. Then each B_i, for $i = 3, \ldots, 6$, touches B_1 and B_2 at their extremity. To be more specific and to confirm that the B_i form a box, we have to check directly with the exact coordinates given in the hypothesis of the theorem.

Finally, we have to prove, with the assumptions of the theorem, that there is a point inside the box B for which the algorithm for the Tetrabrot converges. For this, we remark that z_2^* is between R_1 and R_2. Moreover, $\mathrm{Re}(z_1^*) > \mathrm{Re}(z_2^*)$; then we must have in the set $A := \{x + y i_1 \in \mathbb{C}_1 : x = \mathrm{Re}(z_1^*)$ and $0.0728 < y < 0.0803\}$ a point $z^* \in \mathcal{M}_1$. If not, by hypothesis, the Mandelbrot set would not be connected because z_2^* would be separated from z_1^* by $F_1 \cup A$ since $(F_1 \cup A) \cap \mathcal{M}_1 = \emptyset$.

Now, let

$$y^* := \frac{\mathrm{Im}(z_1^*) - \mathrm{Im}(z^*)}{2}$$

where z_1^*, $z^* \in \mathcal{M}_1$. We note that $z_1^* - y^* i_1 = z^* + y^* i_1$ and $y^* \in [-1, 1] \subset [-m, m]$. Then by Lemma 3, $z_1^* - y^* i_1 - y^* i_2 \in \mathcal{T}$ because $z_1^* - y^* i_1 - y^* i_2 = z^* + y^* i_1 - y^* i_2 \in [(\mathcal{M}_1 - y^* i_1) \cap (\mathcal{M}_1 + y^* i_1)] - y^* i_2 \subset \mathcal{T}$. Moreover, $z_1^* - y^* i_1 - y^* i_2$ is inside the "box of divergence" because $y_1 < y^* < y_2$ and z_1^* is inside the rectangle formed by $L_3 \cup L_4 \cup L_5 \cup L_6$. □

8. CONCLUSION

The last theorem is a good indication that the conjecture is true because the hypothesis about the Mandelbrot set can be approximately confirmed by computers with a high level of precision. To confirm that the conjecture is true, we have two choices — to demonstrate theoretically the hypothesis about the geometry of the Mandelbrot set or to prove more directly that the Tetrabrot is unconnected. If the conjecture is proven to be true, a new question could be to know the cardinality of the family of the connected components. Also, it could be interesting to know whether the "filled-Julia" set associated with points on an unconnected piece of the Tetrabrot have some specific properties such as to be also unconnected. Finally, another pertinent question could be to know the local fractal dimension of the boundary of the Tetrabrot.

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