A Bicomplex Riemann Zeta Function

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May 4, 2002

*Research supported by FIR (UQTR) and CRSNG (Canada).

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The Riemann Zeta Function

The Riemann zeta function is the function of complex variable \( s \), defined in the half-plane \( Re(s) > 1 \) by the convergent series

\[
\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s},
\]

and in the «whole» complex plane \( \mathbb{C} \) by analytic continuation.

Analytic continuation of \( \zeta(s) \)

As shown by Riemann, \( \zeta(s) \) extends to \( \mathbb{C} \) as a meromorphic function with only a simple pole at \( s = 1 \), with residue 1.

A globally convergent series for the Riemann zeta function is given by:

\[
\zeta(s) = \sum_{n=0}^{\infty} \left[ \frac{1}{2^n+1} \sum_{k=0}^{n} \frac{(-1)^k \binom{n}{k}}{(k+1)^{s}} \right], \quad s \in \mathbb{C} \setminus \{1\}.
\]
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**Zeros of \( \zeta(s) \)**

The function \( \zeta(s) \) has zero at the negative even integer \(-2, -4, \ldots\) and one refers to them as the trivial zeros.

**Riemann hypothesis:**

*The nontrivial zeros of \( \zeta(s) \) have real part equal to \( \frac{1}{2} \).*

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**Euler product**

The connection between prime numbers and the zeta function is the celebrated **Euler product**:

\[
\zeta(\sigma) = \prod_{n=1}^{\infty} \frac{1}{1 - \frac{1}{p_n^\sigma}}, \text{ with } \sigma \in \mathbb{R}, \sigma > 1.
\]

Where \( p_1, p_2, \ldots, p_n, \ldots \) is the ascending sequence of primes numbers.
Riemann’s fundamental idea is to extend Euler’s formula to a complex variable. Thus he sets:

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{n=1}^{\infty} \frac{1}{1 - \frac{1}{p_n^s}}
\]

for every complex number \( s \) with \( \text{Re}(s) > 1 \).

**Remark:**

\[ p_n^* := e^{s \cdot \ln(p_n)} = \cos(s \cdot \ln(p_n)) + i \cdot \sin(s \cdot \ln(p_n)) \]

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**Bicomplex Numbers**

In 1892, in search for and development of special algebras, Corrado Segre (1860-1924) published a paper in which he treated an infinite set of algebras whose elements he called bicomplex numbers, tricomplex numbers, ..., \( n \)-complex numbers.

We define **bicomplex numbers** as follows:

\[
\mathbb{C}_2 := \{a + bi_1 + ci_2 + dj : i_1^2 = i_2^2 = -1, \ j^2 = 1\}
\]

where \( i_2 j = ji_2 = -i_1, \ i_1 j = ji_1 = -i_2, \ i_2 i_1 = i_1 i_2 = j \)

and \( a, b, c, d \in \mathbb{R} \).
We remark that we can write a bicomplex number $a + bi_1 + ci_2 + dj$ as:

\[(a + bi_1) + (c + di_1)i_2 = z_1 + z_2i_2\]

where $z_1, z_2 \in \mathbb{C}_1 := \{x + yi_1 : i_1^2 = -1\}$. Thus, $\mathbb{C}_2$ can be viewed as the complexification of $\mathbb{C}_1$ and a bicomplex number can be seen as an element of $\mathbb{C}_2$. Moreover, $\mathbb{C}_2$ is a commutative unitary ring with the following characterization for the noninvertible elements.

Let $w = z_1 + z_2i_2 \in \mathbb{C}_2$. Then $w$ is noninvertible if and only if:

\[z_1^2 + z_2^2 = 0.\]

**Bicomplex Analysis**

It is also possible to define differentiability of a function at a point of $\mathbb{C}_2$:

**Definition 1** Let $U$ be an open set of $\mathbb{C}_2$ and $w_0 \in U$. Then, $f : U \subseteq \mathbb{C}_2 \rightarrow \mathbb{C}_2$ is said to be $\mathbb{C}_2$-differentiable at $w_0$ with derivative equal to $f'(w_0) \in \mathbb{C}_2$ if

\[
\lim_{(w \to w_0) \text{ in } U} \frac{f(w) - f(w_0)}{w - w_0} = f'(w_0).
\]

We will also say that the function $f$ is $\mathbb{C}_2$-holomorphic on an open set $U$ iff $f$ is $\mathbb{C}_2$-differentiable at each point of $U$. 
As we saw, a bicomplex number can be seen as an element of $\mathbb{C}^2$, so a function $f(z_1 + z_2i_2) = f_1(z_1, z_2) + f_2(z_1, z_2)i_2$ of $\mathbb{C}_2$ can be seen as a mapping $f(z_1, z_2) = (f_1(z_1, z_2), f_2(z_1, z_2))$ of $\mathbb{C}^2$. Here we have a characterization of such mappings:

**Theorem 1** Let $U$ be an open set and $f : U \subseteq \mathbb{C}_2 \rightarrow \mathbb{C}_2$. Let also $f(z_1 + z_2i_2) = f_1(z_1, z_2) + f_2(z_1, z_2)i_2$. Then $f$ is $\mathbb{C}_2$-holomorphic on $U$ iff:

$f_1$ and $f_2$ are holomorphic in $z_1$ and $z_2$ and,

$$\frac{\partial f_1}{\partial z_1} = \frac{\partial f_2}{\partial z_2} \quad \text{and} \quad \frac{\partial f_2}{\partial z_1} = -\frac{\partial f_1}{\partial z_2} \quad \text{on} \quad U.$$ 

Now, it is natural to define for $\mathbb{C}^2$ the following class of mappings:

**Definition 2** The class of $T$-holomorphic mappings on a open set $U \subseteq \mathbb{C}^2$ is defined as follows:

$$TH(U) := \{f : U \subseteq \mathbb{C}^2 \rightarrow \mathbb{C}^2 \mid f \in H(U) \} \quad \text{and} \quad \frac{\partial f_1}{\partial z_1} = \frac{\partial f_2}{\partial z_2}, \frac{\partial f_2}{\partial z_1} = -\frac{\partial f_1}{\partial z_2} \quad \text{on} \quad U.$$ 

It is the subclass of holomorphic mappings of $\mathbb{C}^2$ satisfying the complexified Cauchy-Riemann equations.
The idempotent basis

We remark that \( f \in TH(U) \) iff \( f \) is \( \mathbb{C}_2 \)-holomorphic on \( U \). It is also important to know that every bicomplex number \( z_1 + z_2i_2 \) has the following unique idempotent representation:

\[
z_1 + z_2i_2 = (z_1 - z_2i_1)e_1 + (z_1 + z_2i_1)e_2
\]

where \( e_1 = \frac{1+j}{2} \) and \( e_2 = \frac{1-j}{2} \).

This representation is very useful because: addition, multiplication and division can be done term-by-term. Also, an element will be noninvertible iff \( z_1 - z_2i_1 = 0 \) or \( z_1 + z_2i_1 = 0 \).

The notion of **holomorphicity** can also be seen with this kind of notation. For this we need the following definition:

**Definition 3** We say that \( X \subseteq \mathbb{C}_2 \) is a \( \mathbb{C}_2 \)-cartesian set determined by \( X_1 \) and \( X_2 \) if

\[
X = X_1 \times_e X_2 := \{ z_1 + z_2i_2 \in \mathbb{C}_2 : z_1 + z_2i_2 = w_1e_1 + w_2e_2, (w_1, w_2) \in X_1 \times X_2 \}.
\]

**Remark**:

If \( X_1 \) and \( X_2 \) are domains of \( \mathbb{C}_1 \) then \( X_1 \times_e X_2 \) is also a domain of \( \mathbb{C}_2 \).
Now, it is possible to state the following striking theorems:

**Theorem 2** If $f_{e_1} : X_1 \rightarrow \mathbb{C}_1$ and $f_{e_2} : X_2 \rightarrow \mathbb{C}_1$ are holomorphic functions of $\mathbb{C}_1$ on the domains $X_1$ and $X_2$ respectively, then the function $f : X_1 \times_e X_2 \rightarrow \mathbb{C}_2$ defined as

$$f(z_1 + z_2i_2) = f_{e_1}(z_1 - z_2i_1)e_1 + f_{e_2}(z_1 + z_2i_1)e_2,$$

$$\forall z_1 + z_2i_2 \in X_1 \times_e X_2$$

is \textit{T-holomorphic} on the domain $X_1 \times_e X_2$.

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Let $n \in \mathbb{N}\setminus\{0\}$ and $w = z_1 + z_2i_2 \in \mathbb{C}_2$. We define

$$n^w := e^{w \cdot \ln(n)}$$

where

$$e^{z_1 + z_2i_2} := e^{z_1 \cdot e^{z_2i_2}} \text{ and } e^{z_2i_2} := \cos(z_2) + i_2 \sin(z_2).$$

Hence,

$$n^{z_1 + z_2i_2} = e^{z_1 \cdot \ln(n)} \cdot [\cos(z_2 \cdot \ln(n)) + i_2 \sin(z_2 \cdot \ln(n))].$$
**Remarks:**

- $e^{w_1+w_2} = e^{w_1} \cdot e^{w_2}$ $\forall w_1, w_2 \in \mathbb{C}_2$.
- $e^w$ is invertible $\forall w \in \mathbb{C}_2$.
- $e^{z_1+z_2i_2} = (e^{z_1-z_2i_1})e_1 + (e^{z_1+z_2i_1})e_2$ $\forall z_1 + z_2i_2 \in \mathbb{C}_2$.

**Definition 4** Let $w = z_1 + z_2i_2 \in \mathbb{C}_2$ with $Re(z_1) > 1$ and $|Im(z_2)| < Re(z_1) - 1$. We define a **bicomplex Riemann zeta function** $\zeta(w)$ by the following convergent series:

$$
\zeta(w) = \sum_{n=1}^{\infty} \frac{1}{nw}.
$$

The last definition can be well justified by the following theorem:

**Theorem 3** Let $w = z_1 + z_2i_2 \in \mathbb{C}_2$ with $Re(z_1 - z_2i_1) > 1$ and $Re(z_1 + z_2i_1) > 1$.

Then

$$
\zeta(w) = \left[ \sum_{n=1}^{\infty} \frac{1}{n(z_1 - z_2i_1)} \right] e_1 + \left[ \sum_{n=1}^{\infty} \frac{1}{n(z_1 + z_2i_1)} \right] e_2.
$$

Moreover,

$$
\{ w \in \mathbb{C}_2 \mid Re(z_1 - z_2i_1) > 1 \text{ and } Re(z_1 + z_2i_1) > 1 \}
$$

$$
= \{ w \in \mathbb{C}_2 \mid Re(z_1) > 1 \text{ and } |Im(z_2)| < Re(z_1) - 1 \}. 
$$
We will now determine the whole domain of existence of our bicomplex Riemann zeta function. In fact, if $\mathcal{O}_2$ is the set of noninvertible elements in $\mathbb{C}_2$, we extend $\zeta(w)$ as follow:

$$\zeta(w) := \zeta(z_1 - z_2i_1)e_1 + \zeta(z_1 + z_2i_1)e_2$$

on the set $\mathbb{C}_2\{1 + \mathcal{O}_2\}$.

**Remarks:**

- $w \in 1 + \mathcal{O}_2 \Leftrightarrow z_1 - z_2i_1 = 1$ or $z_1 + z_2i_1 = 1$.

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- The set $\mathbb{C}_2\{1 + \mathcal{O}_2\}$ is open and connected in $\mathbb{C}^2$.
- By Theorem 2, $\zeta(w)$ is $T$-holomorphic on $\mathbb{C}_2\{1 + \mathcal{O}_2\}$.
- By the identity theorem of $\mathbb{C}^2$, our analytic continuation of $\zeta(w)$ is unique.
- Let $w_0 \in 1 + \mathcal{O}_2$ then

$$\lim_{\substack{w \to w_0 \\ (w \notin 1 + \mathcal{O}_2)}} |\zeta(w)| = \infty.$$

Hence, the domain $\mathbb{C}_2\{1 + \mathcal{O}_2\}$ is the best possible.
The Zeros of $\zeta(w)$

Let $w = z_1 + z_2 i_2 \in \mathbb{C}_2 \setminus \{1 + \mathcal{O}_2\}$. Then,

$$\zeta(w) = 0 \iff \zeta(z_1 - z_2 i_1) = 0 \text{ and } \zeta(z_1 + z_2 i_1) = 0$$

Hence, from the trivial zeros of $\zeta(s)$ we can obtain trivial zeros for $\zeta(w)$. More specifically, the set $z_1 + z_2 i_2 \in \mathbb{C}_2$ such that

$$z_1 + z_2 i_2 = (-n_1 - n_2) + (-n_1 + n_2)j$$

where $n_1, n_2 \in \mathbb{N} \setminus \{0\}$ will be defined as the set of the **trivial zeros** for the bicomplex Riemann zeta function.

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The Bicomplex Riemann Hypothesis

Moreover, we can establish a bicomplex Riemann hypothesis for $\zeta(w)$ equivalent to the Riemann hypothesis for $\zeta(s)$:

**Conjecture 1** Let $w = z_1 + z_2 i_2 \in \mathbb{C}_2 \setminus \{1 + \mathcal{O}_2\}$. If $w$ is a nontrivial zeros of the bicomplex Riemann zeta function then:

$$(Re(z_1), Im(z_2)) = \left(\frac{1}{2}, 0\right)$$

or $$(Re(z_1), Im(z_2)) = \left(\frac{1}{4} - n, \pm \left(\frac{1}{4} + n\right)\right)$$

where $n \in \mathbb{N} \setminus \{0\}$. 
The Bicomplex Infinite Products

In the complex plane, an infinite product is said to converge if and only if at most a finite number of the factors are zero, and if the partial products formed by the nonvanishing factors tend to a finite limit which is different from zero. In the bicomplex case we have to pay attention to the divisors of zero.

**Definition 5** A bicomplex infinite product is said to converge if and only if at most a finite number of the factors are noninvertible, and if the partial products formed by the invertible factors tend to a finite limit which is invertible.

**Theorem 4** Let $w_n = z_{1,n} + z_{2,n}i_2 \in \mathbb{C}_2 \setminus \mathbb{O}_2$ be a sequence of invertible bicomplex numbers. Then, $\prod_{n=1}^{\infty} w_n$ converges if and only if

$$\prod_{n=1}^{\infty} (z_{1,n} - z_{2,n}i_1) \text{ and } \prod_{n=1}^{\infty} (z_{1,n} + z_{2,n}i_1) \text{ converge.}$$

Moreover, in the case of convergence we obtain:

$$\prod_{n=1}^{\infty} w_n = \prod_{n=1}^{\infty} (z_{1,n} - z_{2,n}i_1)e_1 + \prod_{n=1}^{\infty} (z_{1,n} + z_{2,n}i_1)e_2.$$
The Bicomplex Euler Product

Using the last theorem, we are able to establish a bicomplex Euler product:

**Theorem 5** Let $w = z_1 + z_2 i_2 \in \mathbb{C}_2$ with $Re(z_1) > 1$ and $|Im(z_2)| < Re(z_1) - 1$. Then:

$$\zeta(w) = \sum_{n=1}^{\infty} \frac{1}{n^w} = \prod_{n=1}^{\infty} \frac{1}{1 - \frac{1}{p_n^w}}$$

Where $p_1, p_2, \ldots, p_n, \ldots$ is the ascending sequence of primes numbers.