Tricomplex Dynamics and generalized Mandelbrot sets

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Introduction

- In 1982, A. Douady and J. H. Hubbard studied dynamical systems generated by iterations of the quadratic polynomial $z^2 + c$. One of the main result of their work was the proof that the well-known Mandelbrot set for complex numbers is a connected set.
- In 1990, P. Senn suggested to define the Mandelbrot set for another set of numbers : the hyperbolic numbers (also called duplex numbers). He remarked that the Mandelbrot set for this number structure seemed to be a square. Four years later, a proof of this statement was giving by W. Metzler.
- In 2000, D. Rochon [4] used the bicomplex numbers set $\mathbb{M}(2)$ to give a 4D definition of the so-called Mandelbrot set and made 3D slices to get the *Tetrabrot*.
- Recently, Multibrot sets has been obtained in the frame of the tricomplex numbers. This talk is based on the article [3].

Plan

1. Preliminaries
   - Bicomplex Numbers
   - Tricomplex numbers
   - Important remarks
2. Multibrot sets
3. Conclusion

Bicomplex numbers

Definition 1 ($\mathbb{M}(2)$ or $\mathbb{C}_2$ space)

Let $z_1 = x_1 + x_2\mathbf{i_1}$, $z_2 = x_3 + x_4\mathbf{i_1}$ be two complex numbers $\mathbb{M}(1) \simeq \mathbb{C}$ with $\mathbf{i_1}^2 = -1$. A bicomplex number ζ is defined as

$$\zeta = z_1 + z_2\mathbf{i_2} \tag{1}$$

where $\mathbf{i_2}^2 = -1$.

Various representations :

- In terms of four real numbers: $\zeta = x_1 + x_2\mathbf{i_1} + x_3\mathbf{i_2} + x_4\mathbf{j_1}$
- In terms of two idempotent elements:

$$\zeta = (z_1 - z_2\mathbf{i_1})\gamma_1 + (z_1 + z_2\mathbf{i_1})\overline{\gamma}_1$$

where $\gamma_1 = \frac{1+\mathbf{j_1}}{2}$ and $\overline{\gamma}_1 = \frac{1-\mathbf{j_1}}{2}$.

Go to Table

Operations on Bicomplex numbers

Let $\zeta_1 = z_1 + z_2\mathbf{i_2}$ and $\zeta_2 = z_3 + z_4\mathbf{i_2}$.

1) Equality : $\zeta_1 = \zeta_2 \iff z_1 = z_3$ and $z_2 = z_4$.
2) Addition : $\zeta_1 + \zeta_2 := (z_1 + z_3) + (z_2 + z_4)\mathbf{i_2}$.
3) Multiplication : $\zeta_1 \cdot \zeta_2 := (z_1 z_3 - z_2 z_4) + (z_2 z_3 + z_1 z_4)\mathbf{i_2}$.
4) Modulus : $\|\zeta_1\|_2 := \sqrt{|z_1|^2 + |z_2|^2} = \sqrt{\sum_{i=1}^{4} x_i^2}$
5) Cartesian subset: if $X \subset \mathbb{M}(2)$, $X_1 \subset \mathbb{M}(1)$ and $X_2 \subset \mathbb{M}(1)$, then

$$X = X_1 \times_{\gamma_1} X_2 := \{\zeta \in X \, : \, \zeta = u_1\gamma_1 + u_2\overline{\gamma}_1 \, , \; u_1 \in X_1 \text{ and } u_2 \in X_2\} \, .$$

Remark:

- $(\mathbb{M}(2) \, , + \, , \cdot)$ forms a commutative ring with unity and zero divisors.
- $(\mathbb{M}(2) \, , + \, , \cdot \, , \| \cdot \|_2)$ forms a Banach space.

Tricomplex numbers

Definition 2 ($\mathbb{M}(3)$ or $\mathbb{C}_3$ space)

Let $\zeta_1 = z_1 + z_2\mathbf{i_2}$, $\zeta_2 = z_3 + z_4\mathbf{i_2}$ be two bicomplex numbers. A tricomplex number η is defined as

$$\eta = \zeta_1 + \zeta_2\mathbf{i_3} \tag{2}$$

where $\mathbf{i_3}^2 = -1$.

Various representations :

- In terms of four complex numbers: $\eta = z_1 + z_2\mathbf{i_2} + z_3\mathbf{i_3} + z_4\mathbf{j_3}$
- In terms of eight real numbers:

$$\eta = x_1 + x_2\mathbf{i_1} + x_3\mathbf{i_2} + x_4\mathbf{i_3} + x_5\mathbf{i_4} + x_6\mathbf{j_1} + x_7\mathbf{j_2} + x_8\mathbf{j_3}$$

Go to Table

Tricomplex numbers

Various representations (continuing):

- In terms of two idempotent elements:

$$\eta = (\zeta_1 - \zeta_2 \mathbf{i_2})\gamma_2 + (\zeta_1 + \zeta_2 \mathbf{i_2})\overline{\gamma}_2$$

where $\gamma_2 = \frac{1+\mathbf{j_3}}{2}$ and $\overline{\gamma}_2 = \frac{1-\mathbf{j_3}}{2}$.

- In terms of four idempotent elements:

$$\eta = w_1 \cdot \gamma_1\gamma_2 + w_2 \cdot \gamma_1\overline{\gamma}_2 + w_3 \cdot \overline{\gamma}_1\gamma_2 + w_4 \cdot \overline{\gamma}_1\overline{\gamma}_2$$

where $w_i \in \mathbb{M}(1)$ for $i = 1, 2, 3, 4$ and

$$\begin{aligned} w_1 &= (z_1 + z_4) - (z_2 - z_3)\mathbf{i_1} & \quad w_3 &= (z_1 - z_4) - (z_2 + z_3)\mathbf{i_1} \\ w_2 &= (z_1 + z_4) + (z_2 - z_3)\mathbf{i_1} & \quad w_4 &= (z_1 - z_4) + (z_2 + z_3)\mathbf{i_1}. \end{aligned}$$

Idempotent representation

- The tricomplex numbers γ_2 and $\overline{\gamma}_2$ are called idempotent because

$$\gamma_2^2 = \gamma_2, \quad \overline{\gamma}_2^2 = \overline{\gamma}_2 \quad \text{and} \quad \gamma_2 + \overline{\gamma}_2 = 1.$$

- Moreover, the idempotent representation is unique.
- The set of non-invertible elements is denoted by $\mathcal{NC}$ and is characterized as follows

$$\mathcal{NC} := \left\{\eta = \zeta_1 + \zeta_2 \mathbf{i_3} \in \mathbb{M}(3) \ : \ \zeta_1^2 + \zeta_2^2 = 0\right\}.$$

Operations on Tricomplex Numbers

Let $\eta_1 = \zeta_1 + \zeta_2\mathbf{i_2}$ and $\eta_2 = \zeta_3 + \zeta_4\mathbf{i_2}$.

1) Equality: $\eta_1 = \zeta_2 \iff \zeta_1 = \zeta_3$ and $\zeta_2 = \zeta_4$.
2) Addition: $\eta_1 + \eta_2 := (\zeta_1 + \zeta_3) + (\zeta_2 + \zeta_4)\mathbf{i_2}$.
3) Multiplication: $\eta_1 \cdot \eta_2 := (\zeta_1\zeta_3 - \zeta_2\zeta_4) + (\zeta_2\zeta_3 + \zeta_1\zeta_4)\mathbf{i_2}$.
4) Modulus : $\|\eta_1\|_3 := \sqrt{\|\zeta_1\|_2^2 + \|\zeta_2\|_2^2} = \sqrt{\sum_{i=1}^{4} |z_i|^2} = \sqrt{\sum_{i=1}^{8} x_i^2}$
5) Cartesian subset: if $X \subset \mathbb{M}(3)$, $X_1 \subset \mathbb{M}(2)$ and $X_2 \subset \mathbb{M}(2)$, then

$$X = X_1 \times_{\gamma_2} X_2 := \{\eta \in X \,|\, \eta = u_1\gamma_2 + u_2\overline{\gamma}_2\,,\; u_1 \in X_1 \text{ and } u_2 \in X_2\}\,.$$

Remark:

- $(\mathbb{M}(3)\,,+\,,\cdot)$ forms a commutative ring with unity and zero divisors.
- $(\mathbb{M}(3)\,,+\,,\cdot\,,\|\cdot\|_3)$ forms a Banach space.
- Since $X_1 \subset \mathbb{M}(2)$ and $X_2 \subset \mathbb{M}(2)$, X can be expressed as a cartesian product of four subsets of $\mathbb{M}(1) \simeq \mathbb{C}$.

Operations on Tricomplex numbers (continued)

Theorem 3

Let $\eta_1 = \zeta_1 + \zeta_2 \mathbf{i_3}$, *and* $\eta_2 = \zeta_3 + \zeta_4 \mathbf{i_3}$ *be two tricomplex numbers, and* $\eta_1 = u_1\gamma_2 + u_2\overline{\gamma}_2$ *and* $\eta_2 = u_3\gamma_2 + u_4\overline{\gamma}_2$ *be the idempotent representation of* η_1 *and* η_2. *Then,*

1. $\eta_1 + \eta_2 = (u_1 + u_3)\gamma_2 + (u_2 + u_4)\overline{\gamma}_2$;
2. $\eta_1 \cdot \eta_2 = (u_1 \cdot u_3)\gamma_2 + (u_2 \cdot u_4)\overline{\gamma}_2$;
3. $\eta_1^m = u_1^m\gamma_2 + u_2^m\overline{\gamma}_2 \ \forall m \in \mathbb{N}$.

- The idempotent representation allows to sum and multiply terms by terms.

Some viewpoints

Bicomplex numbers:

- as a pair of complex variables (z_1, z_2);
- as a quadruple of real numbers (x_1, x_2, x_3, x_4).

Tricomplex numbers:

- as a pair of bicomplex numbers (ζ_1, ζ_2);
- as a quadruple of complex numbers (z_1, z_2, z_3, z_4);
- as a octuple of real numbers $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$.

So, with bicomplex and tricomplex numbers, we can divide vectors in a certain way.

Subsets of $\mathbb{M}(3)$

Definition 4 (Open and close discus)

Let $\alpha = \alpha_1 + \alpha_2 \mathbf{i_3} \in \mathbb{M}(3)$ and set $r_2 \geq r_1 > 0$.

1. The open discus is the set

$$D_3(\alpha; r_1, r_2) := \{\eta \in \mathbb{M}(3) \,:\, \eta = \zeta_1\gamma_2 + \zeta_2\overline{\gamma}_2, \\ \|\zeta_1 - (\alpha_1 - \alpha_2\mathbf{i_2})\|_2 < r_1 \\ \text{and } \|\zeta_2 - (\alpha_1 + \alpha_2\mathbf{i_2})\|_2 < r_2\}\,. \tag{3}$$

2. The close discus is the set

$$\overline{D_3}(\alpha; r_1, r_2) := \{\eta \in \mathbb{M}(3) \,:\, \eta = \zeta_1\gamma_2 + \zeta_2\overline{\gamma}_2, \\ \|\zeta_1 - (\alpha_1 - \alpha_2\mathbf{i_2})\|_2 \leq r_1 \\ \text{and } \|\zeta_2 - (\alpha_1 + \alpha_2\mathbf{i_2})\|_2 \leq r_2\}\,. \tag{4}$$

Subsets of $\mathbb{M}(3)$ (continued)

Definition 5

Let $\mathbf{i_k} \in \{\mathbf{i_1}, \mathbf{i_2}, \mathbf{i_3}, \mathbf{i_4}\}$ and $\mathbf{j_k} \in \{\mathbf{j_1}, \mathbf{j_2}, \mathbf{j_3}\}$. We define

$$\mathbb{C}(\mathbf{i_k}) := \{\eta = x_0 + x_1\mathbf{i_k} \,:\, x_0, x_1 \in \mathbb{R}\}$$

and

$$\mathbb{D}(\mathbf{j_k}) := \{x_0 + x_1\mathbf{j_k} \,:\, x_0, x_1 \in \mathbb{R}\}.$$

- $\mathbb{C}(\mathbf{i_k})$ is a subset of $\mathbb{M}(3)$ for $k \in \{1, 2, 3, 4\}$. They are all isomorphic to $\mathbb{C}$. Notice that $\mathbb{C}(\mathbf{i_1}) = \mathbb{M}(1)$.
- $\mathbb{D}(\mathbf{j_k})$ is a subset of $\mathbb{M}(3)$ and is isomorphic to the set of hyperbolic numbers $\mathbb{D}$ for $k \in \{1, 2, 3\}$.

Subsets of $\mathbb{M}(3)$ (continued)

Definition 6

Let $\mathbf{i_k}, \mathbf{i_l} \in \{1, \mathbf{i_1}, \mathbf{i_2}, \mathbf{i_3}, \mathbf{i_4}, \mathbf{j_1}, \mathbf{j_2}, \mathbf{j_3}\}$ where $\mathbf{i_k} \neq \mathbf{i_l}$. The first subset is

$$\mathbb{M}(\mathbf{i_k}, \mathbf{i_l}) := \{x_1 + x_2\mathbf{i_k} + x_3\mathbf{i_l} + x_4\mathbf{i_k}\mathbf{i_l} \ : \ x_i \in \mathbb{R}, i = 1, \ldots, 4\}. \quad (5)$$

Definition 7

Let $\mathbf{i_k}, \mathbf{i_l}, \mathbf{i_m} \in \{\mathbf{i_1}, \mathbf{i_2}, \mathbf{i_3}, \mathbf{i_4}, \mathbf{j_1}, \mathbf{j_2}, \mathbf{j_3}\}$ with $\mathbf{i_k} \neq \mathbf{i_l}$, $\mathbf{i_k} \neq \mathbf{i_m}$ and $\mathbf{i_l} \neq \mathbf{i_m}$. The second subset is

$$\mathbb{M}(\mathbf{i_k}, \mathbf{i_l}, \mathbf{i_m}) := \{x_1\mathbf{i_k} + x_2\mathbf{i_l} + x_3\mathbf{i_m} + x_4\mathbf{i_k}\mathbf{i_l}\mathbf{i_m} \ : \ x_i \in \mathbb{R}, i = 1, \ldots, 4\}. \quad (6)$$

- The set $\mathbb{M}(\mathbf{i_k}, \mathbf{i_l})$ is closed under addition and multiplication of tricomplex numbers. Also, $\mathbb{M}(\mathbf{i_k}, \mathbf{i_l}) \simeq \mathbb{M}(2)$ except for the *biduplex* sets $\mathbb{M}(\mathbf{j_1}, \mathbf{j_2})$, $\mathbb{M}(\mathbf{j_1}, \mathbf{j_3})$ and $\mathbb{M}(\mathbf{j_2}, \mathbf{j_3})$.
- The set $\mathbb{M}(\mathbf{i_k}, \mathbf{i_l}, \mathbf{i_m})$ is closed under addition, but not closed under multiplication in general.

Subsets of $\mathbb{M}(3)$ (continued)

Definition 8

Let $\mathbf{i_k}, \mathbf{i_l}, \mathbf{i_m} \in \{1, \mathbf{i_1}, \mathbf{i_2}, \mathbf{i_3}, \mathbf{i_4}, \mathbf{j_1}, \mathbf{j_2}, \mathbf{j_3}\}$ with $\mathbf{i_k} \neq \mathbf{i_l}$, $\mathbf{i_k} \neq \mathbf{i_m}$ and $\mathbf{i_l} \neq \mathbf{i_m}$. The third subset is

$$\mathbb{T}(\mathbf{i_m}, \mathbf{i_k}, \mathbf{i_l}) := \{x_1\mathbf{i_k} + x_2\mathbf{i_l} + x_3\mathbf{i_m} \ : \ x_1, x_2, x_3 \in \mathbb{R}\}. \tag{7}$$

- This set is used to make 3D slices in the Tricomplex Multibrot sets.

Plan

Definition of Multibrots in the complex plane

Definition 9

Let $Q_{p,c}(z) = z^p + c$ a polynomial of degree $p \in \mathbb{N} \setminus \{0,1\}$. A *Multibrot* set is the set of complex numbers c for which the sequence $\left\{Q_{p,c}^m(0)\right\}_{m=1}^{\infty}$ is bounded, *i.e.*

$$\mathcal{M}^p = \left\{ c \in \mathbb{C} : \left\{Q_{p,c}^m(0)\right\}_{m=1}^{\infty} \text{ is bounded } \right\}. \tag{8}$$

- If we set $p = 2$, we find the well-known Mandelbrot set.

Properties of Multibrot sets

Theorem 10

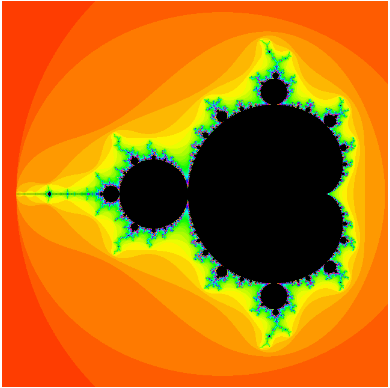
For all complex number c in $\mathcal{M}^p$, we have $|c| \leq 2^{1/(p-1)}$.

Theorem 11

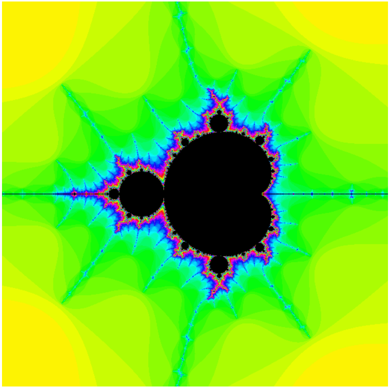
A complex number c is in $\mathcal{M}^p$ if and only if $|Q_{p,c}^m(0)| \leq 2^{1/(p-1)}$ for all natural number $m \geq 1$.

- For an integer $p \geq 2$, the set $\mathcal{M}^p$ is contained in the closed discus in $\mathbb{C}$.
- Theorem 11 gives a method to visualize the Multibrot sets.

Multibrot sets pictured

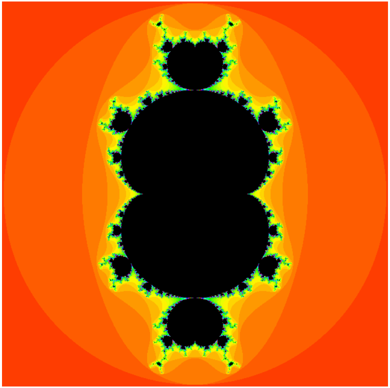


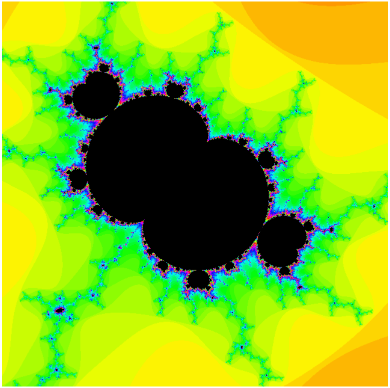
(a) $\mathcal{M}^2$: Mandelbrot set



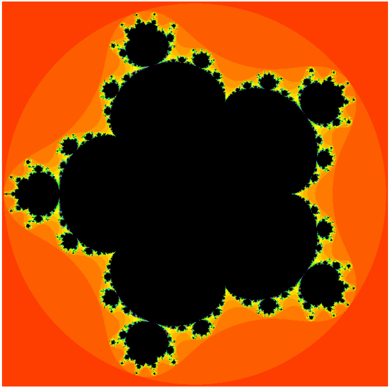
(b) $\mathcal{M}^2$: Zoom in

Multibrot sets pictured

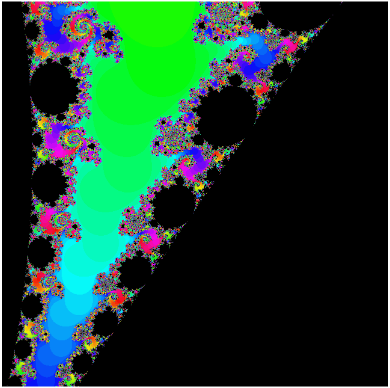

(a) $\mathcal{M}^3$: Mandelbric set


(b) $\mathcal{M}^3$: Zoom in

Multibrot sets pictures



(a) $\mathcal{M}^6$



(b) $\mathcal{M}^6$: Zoom in

Properties of Multibrot sets (continued)

Theorem 12

Let $p \geq 2$ be an integer. Then, $\mathcal{M}^p$ is a connected set.

Lemma 13

Let $c \in \mathcal{M}^p$ with $p \geq 2$ an integer and $c = |c|e^{\mathbf{i}\theta_c}$. Then, $c_k := |c|e^{\mathbf{i}\left(\theta_c + \frac{2k\pi}{p-1}\right)}$ is in $\mathcal{M}^p$ for any $k \in \mathbb{Z}$.

- Theorem 12 is a consequence of a famous result from Douady and Hubbard. The biholomorphic map ϕ_c that conjugate $\mathcal{M}^p$ with the unit circle is called the *Böttcher coordinate*.
- By letting $k = n(p-1) + t$ where $n \in \mathbb{Z}$ and $t \in \{0, 1, \ldots, p-2\}$, we see that $e^{\mathbf{i}\left(\theta_c + \frac{2k\pi}{p-1}\right)} = e^{\mathbf{i}\left(\theta_c + \frac{2t\pi}{p-1}\right)}$. So, the set $\mathcal{M}^p$ has $p-1$ symmetries by the origin.

Special Case: $p = 3$

We know that $\mathcal{M}^2 \cap \mathbb{R} = \left[-2, \frac{1}{4}\right]$. In a similar way, we can characterize the real part of the Multibrot set $\mathcal{M}^3$.

Theorem 14

The set $\mathcal{M}^3$ cross the real axis on the interval $\left[\frac{-2}{3\sqrt{3}}, \frac{2}{3\sqrt{3}}\right]$.

Definition of Hyperbrots

Definition 15

Let $Q_{p,c}(z) = z^p + c$ where $z, c \in \mathbb{D}$ and $p \geq 2$ an integer. The Hyperbrots are defined as the sets

$$\mathcal{H}^p := \left\{ c \in \mathbb{D} \, : \, \left\{ Q_{p,c}^m(0) \right\}_{m=1}^{\infty} \text{ is bounded } \right\}. \tag{9}$$

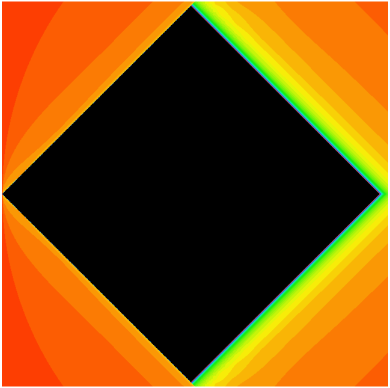
- In the definition, the set $\mathbb{D}$ is the set of hyperbolic numbers, *i.e.*

$$\mathbb{D} := \left\{ x + y\mathbf{j} \, : \, x, y \in \mathbb{R} \text{ and } \mathbf{j}^2 = 1 \right\}.$$

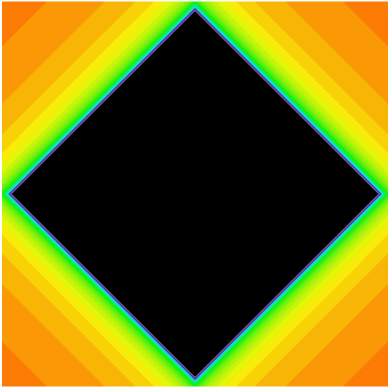
- Addition and multiplication are defined as usual. If $z = x + y\mathbf{j}$ and $w = s + t\mathbf{j}$, then

$$z + w := (x+s) + (y+t)\mathbf{j} \quad \text{and} \quad z \cdot w = (xs + yt) + (xt + ys)\mathbf{j}.$$

Hyperbrot sets pictured

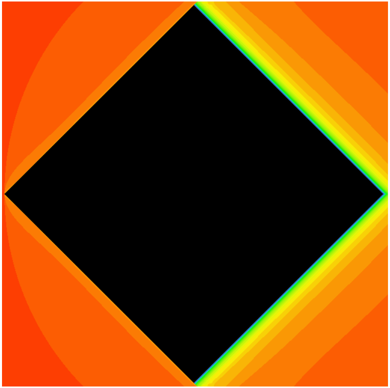


(a) $\mathcal{H}^2$: Mandelbrot set

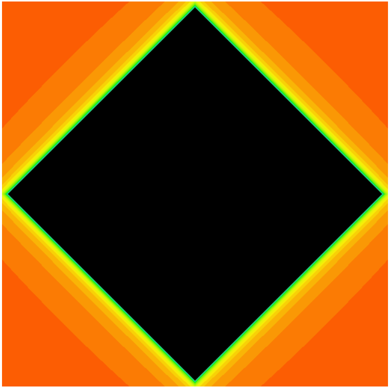


(b) $\mathcal{H}^3$: Hyperbric

Hyperbrot sets pictured



(a) $\mathcal{H}^4$



(b) $\mathcal{H}^9$

Characterization

Metzler proved that the set $\mathcal{H}^2$ is a square with the following characterization:

$$\mathcal{H}^2 = \left\{ c = x + y\mathbf{j} \in \mathbb{D} \, : \, |x - \frac{7}{8}| + |y| \leq \frac{9}{8} \right\}.$$

There is a similar characterization of the set $\mathcal{H}^3$.

Theorem 16

The set $\mathcal{H}^3$ is a square with the following characterization:

$$\mathcal{H}^3 = \left\{ c = x + y\mathbf{j} \in \mathbb{D} \, : \, |x| + |y| \leq \frac{2}{3\sqrt{3}} \right\}$$

Bicomplex Multibrot sets

Definition 17

Let $Q_{p,c}(\zeta) = \zeta^p + c$ where $\zeta, c \in \mathbb{M}(2)$ and $p \geq 2$ is an integer. The bicomplex *Multibrot* set is define as the set

$$\mathcal{M}_2^p := \left\{ c \in \mathbb{M}(2) \ : \ \left\{ Q_{p,c}^m(0) \right\}_{m=1}^{\infty} \text{ is bounded } \right\}.$$

- The set $\mathcal{M}_2^p$ can be expressed as a cartesian product: $\mathcal{M}_2^p = \mathcal{M}^p \times_{\gamma_1} \mathcal{M}^p$.
- It is contained in the closed discuss of radii $2^{1/(p-1)}$.
- It is connected since $\mathcal{M}^p$ is connected.

Tricomplex Multibrot

Definition 18

Let $Q_{p,c}(\eta) = \eta^p + c$ where $\eta, c \in \mathbb{M}(3)$ and $p \geq 2$ an integer. The tricomplex *Multibrot* set is define as the set

$$\mathcal{M}_3^p := \left\{ c \in \mathbb{M}(3) \ : \ \left\{ Q_{p,c}^m(0) \right\}_{m=1}^{\infty} \text{ is bounded } \right\}. \tag{10}$$

Properties of $\mathcal{M}_3^p$

Theorem 19

The tricomplex Multibrot sets for a fixed integer $p \geq 2$ can be expressed as

$$\mathcal{M}_3^p = \mathcal{M}_2^p \times_{\gamma_2} \mathcal{M}_2^p$$

and it is a connected set.

Properties of $\mathcal{M}_3^p$

Theorem 20

Let $\mathcal{M}_3^p$ be the tricomplex Multibrot set for $p \in \mathbb{N} \backslash \{0, 1\}$. Then the following inclusion holds:

$$\mathcal{M}_3^p \subset \overline{D_3}(0, 2^{\frac{1}{p-1}}, 2^{\frac{1}{p-1}}). \tag{11}$$

Theorem 21

A tricomplex number c is in $\mathcal{M}_3^p$ if and only if $|Q_{p,c}^m(0)| \leq 2^{1/(p-1)}$ for all natural number $m \geq 1$.

3D slices of $\mathcal{M}_3^p$

To visualize the Tricomplex multibrot sets, we have to define a principal 3D slice of $\mathcal{M}_3^p$.

$$\mathcal{T}^p := \mathcal{T}^p(\mathbf{i_m}, \mathbf{i_k}, \mathbf{i_l}) = \left\{ c \in \mathbb{T}(\mathbf{i_m}, \mathbf{i_k}, \mathbf{i_l}) \ : \ \left\{ Q_{p,c}^m(0) \right\}_{m=1}^{\infty} \text{ is bounded } \right\}. \tag{12}$$

There are 56 possible 3D principal slices. To attempt a classification, we define a relation $\sim$ between the family of principal 3D slices.

3D slices of $\mathcal{M}_3^p$

Definition 22

Let $\mathcal{T}_1^p(\mathbf{i_m}, \mathbf{i_k}, \mathbf{i_l})$ and $\mathcal{T}_2^p(\mathbf{i_n}, \mathbf{i_q}, \mathbf{i_s})$ be two 3D slices of a tricomplex *Multibrot* set $\mathcal{M}_3^p$ that correspond, respectively, to Q_{p,c_1} and Q_{p,c_2}. Then, $\mathcal{T}_1^p \sim \mathcal{T}_2^p$ if there exists a bijective linear function $\varphi : \mathrm{span}_{\mathbb{R}} \{1, \mathbf{i_m}, \mathbf{i_k}, \mathbf{i_l}\} \to \mathrm{span}_{\mathbb{R}} \{1, \mathbf{i_n}, \mathbf{i_q}, \mathbf{i_s}\}$ such that $(\varphi \circ Q_{p,c_1} \circ \varphi^{-1})(\eta) = Q_{p,c_2}(\eta)\ \forall \eta \in \mathrm{span}_{\mathbb{R}} \{1, \mathbf{i_n}, \mathbf{i_q}, \mathbf{i_s}\}$. In that case, we say that $\mathcal{T}_1^p$ and $\mathcal{T}_2^p$ have the same dynamics.

- If two slices are in relation, then we say that they are symmetrical.
- The relation $\sim$ is an equivalent relation.

Principal slices of $\mathcal{M}_3^3$

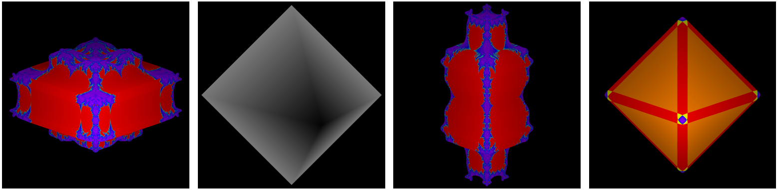
The number of principal 3D slices of the set $\mathcal{M}_3^3$ can be reduced to only four slices !

Theorem 23

There are four principal 3D slices of the tricomplex Multibrot set $\mathcal{M}_3^3$:

- $\mathcal{T}^3(1, \mathbf{i_1}, \mathbf{i_2})$ *called Tetrabric;*
- $\mathcal{T}^3(1, \mathbf{j_1}, \mathbf{j_2})$ *called Perplexbric;*
- $\mathcal{T}^3(1, \mathbf{i_1}, \mathbf{j_1})$ *called Hourglassbric;*
- $\mathcal{T}^3(\mathbf{i_1}, \mathbf{i_2}, \mathbf{i_3})$ *called Metabric.*

Family shooting: cubic $\eta^3 + c$



(a) Tetrabric (b) Perplexbric (c) Hourglassbric (d) Metabric

Special case: Perplexbric

Recall that the perplexbric $\mathcal{P}^3$ is defined as the set

$$\left\{ c = c_1 + c_4\mathbf{j_1} + c_6\mathbf{j_2} \,:\, c_i \in \mathbb{R} \text{ and } \left\{ Q_{3,c}^m(0) \right\}_{n=1}^{\infty} \text{ is bounded} \right\}.$$

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Lemma 24

We have the following characterization of the Perplexbric

$$\mathcal{P}^3 = \bigcup_{y \in \left[\frac{-2}{3\sqrt{3}}, \frac{2}{3\sqrt{3}}\right]} \left\{\left[(\mathcal{H}^3 - y\mathbf{j_1}) \cap (\mathcal{H}^3 + y\mathbf{j_1})\right] + y\mathbf{j_2}\right\}$$

where $\mathcal{H}^3$ is the Hyperbrot generated by the polynomial $z^3 + c$.

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Corollary: The set $\mathcal{P}^3$ is a regular octahedron.

Plan

Futur work

In general, we have some conjectures:

Conjecture 1 (Real intersection)

Let $\mathcal{M}^p$ be the generalized Mandelbrot set for the polynomial $Q_{p,c}(z) = z^p + c$ where $z, c \in \mathbb{C}$ and $p \geq 2$ an integer. Then, we have two cases for the intersection $\mathcal{M}^p \cap \mathbb{R}$:

i. *If p is even, then*

$$\mathcal{M}^p \cap \mathbb{R} = \left[-2^{\frac{1}{p-1}}, (p-1)p^{\frac{-p}{p-1}}\right]; \tag{13}$$

ii. *If p is odd, then*

$$\mathcal{M}^p \cap \mathbb{R} = \left[-(p-1)p^{\frac{-p}{p-1}}, (p-1)p^{\frac{-p}{p-1}}\right]. \tag{14}$$

Futur work

In general, we have some conjectures:

Conjecture 2 (Squares)

The hyperbrots are squares.

- To prove Conjecture 1, we have to find another approach since there is no general formula for the roots of the polynomial $z^p + c$ when $p \geq 5$.
- The second conjecture is a direct consequence of Conjecture 1.

Futur work

In general, what happens with the principal 3D slices of the tricomplex Multibrot sets.

Question

How many principal 3D slices are they in general?

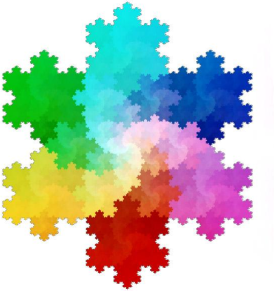
Table of imaginary units

$\cdot$	1	$\mathbf{i_1}$	$\mathbf{i_2}$	$\mathbf{i_3}$	$\mathbf{i_4}$	$\mathbf{j_1}$	$\mathbf{j_2}$	$\mathbf{j_3}$
1	1	$\mathbf{i_1}$	$\mathbf{i_2}$	$\mathbf{i_3}$	$\mathbf{i_4}$	$\mathbf{j_1}$	$\mathbf{j_2}$	$\mathbf{j_3}$
$\mathbf{i_1}$	$\mathbf{i_1}$	$-\mathbf{1}$	$\mathbf{j_1}$	$\mathbf{j_2}$	$-\mathbf{j_3}$	$-\mathbf{i_2}$	$-\mathbf{i_3}$	$\mathbf{i_4}$
$\mathbf{i_2}$	$\mathbf{i_2}$	$\mathbf{j_1}$	$-\mathbf{1}$	$\mathbf{j_3}$	$-\mathbf{j_2}$	$-\mathbf{i_1}$	$\mathbf{i_4}$	$-\mathbf{i_3}$
$\mathbf{i_3}$	$\mathbf{i_3}$	$\mathbf{j_2}$	$\mathbf{j_3}$	$-\mathbf{1}$	$-\mathbf{j_1}$	$\mathbf{i_4}$	$-\mathbf{i_1}$	$-\mathbf{i_2}$
$\mathbf{i_4}$	$\mathbf{i_4}$	$-\mathbf{j_3}$	$-\mathbf{j_2}$	$-\mathbf{j_1}$	$-\mathbf{1}$	$\mathbf{i_3}$	$\mathbf{i_2}$	$\mathbf{i_1}$
$\mathbf{j_1}$	$\mathbf{j_1}$	$-\mathbf{i_2}$	$-\mathbf{i_1}$	$\mathbf{i_4}$	$\mathbf{i_3}$	$\mathbf{1}$	$-\mathbf{j_3}$	$-\mathbf{j_2}$
$\mathbf{j_2}$	$\mathbf{j_2}$	$-\mathbf{i_3}$	$\mathbf{i_4}$	$-\mathbf{i_1}$	$\mathbf{i_2}$	$-\mathbf{j_3}$	$\mathbf{1}$	$-\mathbf{j_1}$
$\mathbf{j_3}$	$\mathbf{j_3}$	$\mathbf{i_4}$	$-\mathbf{i_3}$	$-\mathbf{i_2}$	$\mathbf{i_1}$	$-\mathbf{j_2}$	$-\mathbf{j_1}$	$\mathbf{1}$

Table: Product of tricomplex imaginary units

RetourBi RetourTri

Thanks for your attention!



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