Bicomplex Hilbert Spaces

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- Bicomplex numbers, just like quaternions, are a generalization of complex numbers by means of entities specified by four real numbers. These two number systems, however, are different in two important ways: quaternions, which form a division algebra, are noncommutative, whereas bicomplex numbers are commutative but do not form a division algebra.
- Division algebras do not have zero divisors, that is, nonzero elements whose product is zero. Many believe that any attempt to generalize quantum mechanics to number systems other than complex numbers should retain the division algebra property. Indeed considerable work has been done over the years on quaternionic quantum mechanics.

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- In the past few years, however, it was pointed out that several features of quantum mechanics can be generalized to bicomplex numbers. A generalization of Schrödinger's equation for a particle in one dimension was proposed, and self-adjoint operators were defined on finite-dimensional bicomplex Hilbert spaces. Recently, eigenvalues and eigenfunctions of the bicomplex analogue of the quantum harmonic oscillator Hamiltonian were obtained in full generality.
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- The perspective of generalizing quantum mechanics to bicomplex numbers motivates us in developing further mathematical tools related to finite-dimensional bicomplex Hilbert spaces and operators acting on them.

Definition

Bicomplex numbers are defined as

$$\mathbb{T} := \{z_1 + z_2\mathbf{i_2} \mid z_1, z_2 \in \mathbb{C}(\mathbf{i_1})\} \tag{1}$$

where the imaginary units $\mathbf{i_1}, \mathbf{i_2}$ and $\mathbf{j}$ are governed by the rules: $\mathbf{i_1}^2 = \mathbf{i_2}^2 = -1$, $\mathbf{j}^2 = 1$ and

$$\begin{array}{rcccl} \mathbf{i_1}\mathbf{i_2} & = & \mathbf{i_2}\mathbf{i_1} & = & \mathbf{j}, \\ \mathbf{i_1}\mathbf{j} & = & \mathbf{j}\mathbf{i_1} & = & -\mathbf{i_2}, \\ \mathbf{i_2}\mathbf{j} & = & \mathbf{j}\mathbf{i_2} & = & -\mathbf{i_1}. \end{array} \tag{2}$$

- Note that we define $\mathbb{C}(\mathbf{i}_k) := \{x + y\mathbf{i}_k \mid \mathbf{i}_k^2 = -1 \text{ and } x, y \in \mathbb{R}\}$ for $k = 1, 2$.

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In fact, the bicomplex numbers

$$\mathbb{T} \cong \mathrm{Cl}_{\mathbb{C}}(1,0) \cong \mathrm{Cl}_{\mathbb{C}}(0,1)$$

are *unique* among the complex Clifford algebras in that they are commutative but not division algebra. It is also convenient to write the set of bicomplex numbers as

$$\mathbb{T} := \{w_0 + w_1\mathbf{i_1} + w_2\mathbf{i_2} + w_3\mathbf{j} \mid w_0, w_1, w_2, w_3 \in \mathbb{R}\}. \tag{3}$$

- In particular, if we put $z_1 = x$ and $z_2 = y\mathbf{i_1}$ with $x, y \in \mathbb{R}$ in $z_1 + z_2\mathbf{i_2}$, then we obtain the following subalgebra of hyperbolic numbers, also called duplex numbers:

$$\mathbb{D} := \{x + y\mathbf{j} \mid \mathbf{j}^2 = 1,\ x, y \in \mathbb{R}\} \cong \mathrm{Cl}_{\mathbb{R}}(0,1).$$

- Zero divisors make up the so-called null cone $\mathcal{NC}$. That terminology comes from the fact that when w is written as $z_1 + z_2\mathbf{i_2}$, zero divisors are such that $z_1^2 + z_2^2 = 0$.

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- Complex conjugation plays an important role both for algebraic and geometric properties of $\mathbb{C}$. For bicomplex numbers, there are three possible conjugations. Let $w \in \mathbb{T}$ and $z_1, z_2 \in \mathbb{C}(\mathbf{i_1})$ such that $w = z_1 + z_2\mathbf{i_2}$. Then we define the three conjugations as:

$$w^{\dagger_1} = (z_1 + z_2\mathbf{i_2})^{\dagger_1} := \bar{z}_1 + \bar{z}_2\mathbf{i_2}, \tag{4a}$$

$$w^{\dagger_2} = (z_1 + z_2\mathbf{i_2})^{\dagger_2} := z_1 - z_2\mathbf{i_2}, \tag{4b}$$

$$w^{\dagger_3} = (z_1 + z_2\mathbf{i_2})^{\dagger_3} := \bar{z}_1 - \bar{z}_2\mathbf{i_2}, \tag{4c}$$

where $\bar{z}_k$ is the standard complex conjugate of complex numbers $z_k \in \mathbb{C}(\mathbf{i_1})$.

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We know that the product of a standard complex number with its conjugate gives the square of the Euclidean metric in $\mathbb{R}^2$. The analogs of this, for bicomplex numbers, are the following. Let $z_1, z_2 \in \mathbb{C}(\mathbf{i_1})$ and $w = z_1 + z_2\mathbf{i_2} \in \mathbb{T}$, then we have that:

$$|w|_{\mathbf{i_1}}^2 := w \cdot w^{\dagger_2} \in \mathbb{C}(\mathbf{i_1}), \tag{5a}$$

$$|w|_{\mathbf{i_2}}^2 := w \cdot w^{\dagger_1} \in \mathbb{C}(\mathbf{i_2}), \tag{5b}$$

$$|w|_{\mathbf{j}}^2 := w \cdot w^{\dagger_3} \in \mathbb{D}. \tag{5c}$$

In this talk we will often use the Euclidean $\mathbb{R}^4$ norm defined as

$$|w| := \sqrt{|z_1|^2 + |z_2|^2} = \sqrt{\mathrm{Re}(|w|_{\mathbf{j}}^2)}\,. \tag{6}$$

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$$z_1 + z_2\mathbf{i_2} = (z_1 - z_2\mathbf{i_1})\mathbf{e_1} + (z_1 + z_2\mathbf{i_1})\mathbf{e_2}. \tag{7}$$

where $\mathbf{e_1} = \frac{1+\mathbf{j}}{2}$ and $\mathbf{e_2} = \frac{1-\mathbf{j}}{2}$.

- From this, we can introduce two projection operators

$$P_1 : (z_1 + z_2\mathbf{i_2}) \in \mathbb{T} \mapsto (z_1 + z_2\mathbf{i_2})_{\hat{1}} \in \mathbb{C}(\mathbf{i_1}), \tag{8}$$

$$P_2 : (z_1 + z_2\mathbf{i_2}) \in \mathbb{T} \mapsto (z_1 + z_2\mathbf{i_2})_{\hat{2}} \in \mathbb{C}(\mathbf{i_1}). \tag{9}$$

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- The projection operators P_k have the interesting properties

$$[P_k]^2 = P_k, \qquad P_1\mathbf{e_1} + P_2\mathbf{e_2} = \mathbf{Id}, \qquad k = 1, 2, \tag{10}$$

and for $s, t \in \mathbb{T}$, we have

$$P_k(s + t) = P_k(s) + P_k(t), \quad P_k(s \cdot t) = P_k(s) \cdot P_k(t), \quad k = 1, 2.$$

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- Bicomplex numbers make up a commutative ring. What vector spaces are to fields, modules are to rings. A module defined over the ring $\mathbb{T}$ of bicomplex numbers will be called a $\mathbb{T}$-*module*.

Definition

A *basis* of a $\mathbb{T}$-module is a set of linearly independent elements that generate the module.[a]

[a] The term "basis" here should not be confused with the same word appearing in "idempotent basis".

- A finite-dimensional *free* $\mathbb{T}$-module is a $\mathbb{T}$-module with a finite basis. That is, M is a finite-dimensional free $\mathbb{T}$-module if there exist n linearly independent elements (denoted $|m_l\rangle$) such that any element $|\psi\rangle$ of M can be written as

$$|\psi\rangle = \sum_{l=1}^{n} w_l |m_l\rangle, \tag{11}$$

with $w_l \in \mathbb{T}$. We have used Dirac's notation for elements of M which, following, we will call *kets*.

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- An important subset V of M is the set of all kets for which all w_l in (11) belong to $\mathbb{C}(\mathbf{i_1})$. In other words, V is the set of all $|\psi\rangle$ so that

$$|\psi\rangle = \sum_{l=1}^{n} x_l |m_l\rangle, \qquad x_l \in \mathbb{C}(\mathbf{i_1}). \tag{12}$$

It is easy to see that V is a vector space over the complex numbers, and that any $|\psi\rangle \in \mathbb{T}$ can be decomposed uniquely as

$$|\psi\rangle = \mathbf{e_1}|\psi\rangle_{\widehat{1}} + \mathbf{e_2}|\psi\rangle_{\widehat{2}} = \mathbf{e_1}P_1\left(|\psi\rangle\right) + \mathbf{e_2}P_2\left(|\psi\rangle\right). \tag{13}$$

Here $|\psi\rangle_{\widehat{k}} \in V$ and P_k is a projector from M to V ($k = 1, 2$). One can show that ket projectors and idempotent-basis projectors (denoted with the same symbol) satisfy

$$P_k\left(s|\psi\rangle + t|\phi\rangle\right) = P_k\left(s\right)P_k\left(|\psi\rangle\right) + P_k\left(t\right)P_k\left(|\phi\rangle\right). \tag{14}$$

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We now define two important subsets of M.

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For $k = 1, 2$, $V_k := \left\{\mathbf{e_k} \sum_{l=1}^{n} x_l |m_l\rangle \mid x_l \in \mathbb{C}(\mathbf{i_1})\right\}$. Or succinctly, $V_k := \mathbf{e_k} V$.

- Clearly, V_k is an n-dimensional vector space over $\mathbb{C}(\mathbf{i_1})$, isomorphic to V and with $\mathbf{e_k}|m_l\rangle$ as basis elements. All three vector spaces V, V_1 and V_2 are useful.
- Although V depends on the choice of basis $\{|m_l\rangle\}$, V_1 and V_2 do not. This comes from the fact that any element of V_1 (for instance) can be written as $\mathbf{e_1}|\psi\rangle$, with $|\psi\rangle$ in M. Clearly, this makes no reference to any specific basis.

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The module M, defined over the ring $\mathbb{T}$, has n dimensions. We now show that the set of elements of M can also be viewed as a $2n$-dimensional vector space over $\mathbb{C}(\mathbf{i_1})$, which we shall call M'. To see this, we write in the idempotent basis the coefficients w_l of a general element of M. We get

$$|\psi\rangle = \sum_{l=1}^{n}(\mathbf{e_1} w_{l\hat{1}} + \mathbf{e_2} w_{l\hat{2}})|m_l\rangle = \sum_{l=1}^{n} w_{l\hat{1}}\mathbf{e_1}|m_l\rangle + \sum_{l=1}^{n} w_{l\hat{2}}\mathbf{e_2}|m_l\rangle. \quad (15)$$

It is not difficult to show that the $2n$ elements $\mathbf{e_1}|m_l\rangle$ and $\mathbf{e_2}|m_l\rangle$ ($l = 1 \ldots n$) are linearly independent over $\mathbb{C}(\mathbf{i_1})$. This proves our claim and, moreover, proves

Theorem

$M' = V_1 \oplus V_2$.

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It is not difficult to show that the $2n$ elements $\mathbf{e_1}|m_l\rangle$ and $\mathbf{e_2}|m_l\rangle$ $(l = 1 \ldots n)$ are linearly independent over $\mathbb{C}(\mathbf{i_1})$. This proves our claim and, moreover, proves

Theorem

$M' = V_1 \oplus V_2$.

It is well known that all bases of a finite-dimensional vector space have the same number of elements. This, however, is not true in general for modules. However, for $\mathbb{T}$-modules we have

Theorem

Let M be a finite-dimensional free $\mathbb{T}$-module. Then all bases of M have the same number of elements.

With the projections P_k defined with respect to the $|m_l\rangle$, it is obvious that $P_k(|m_l\rangle) = |m_l\rangle$ $(k = 1, 2)$. This is a direct consequence of the identity $|m_l\rangle = \mathbf{e_1}|m_l\rangle + \mathbf{e_2}|m_l\rangle$. Hence $\{P_k(|m_l\rangle) \mid l = 1 \dots n\}$ is a basis of V. It turns out that the projection of any basis of M is a basis of V.

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- A *bicomplex linear operator* A is a mapping from M to M such that, for any $s, t \in \mathbb{T}$ and any $|\psi\rangle, |\phi\rangle \in M$,

$$A(s|\psi\rangle + t|\phi\rangle) = sA|\psi\rangle + tA|\phi\rangle. \tag{16}$$

A bicomplex linear operator A can always be written as

$$A = \mathbf{e_1}A_{\widehat{1}} + \mathbf{e_2}A_{\widehat{2}} = \mathbf{e_1}P_1(A) + \mathbf{e_2}P_2(A). \tag{17}$$

By definition, we have that ($k = 1, 2$)

$$P_k(A)|\psi\rangle = P_k(A|\psi\rangle) \qquad \forall|\psi\rangle \in M. \tag{18}$$

Clearly one can write

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Definition

A ket $|\psi\rangle$ belongs to the null cone if either $|\psi\rangle_{\widehat{1}} = 0$ or $|\psi\rangle_{\widehat{2}} = 0$. A linear operator A belongs to the null cone if either $A_{\widehat{1}} = 0$ or $A_{\widehat{2}} = 0$.

- The following definition adapts to modules the concepts of eigenvector and eigenvalue, most useful in vector space theory.

Definition

Let $A : M \to M$ be a bicomplex linear operator and let

$$A|\psi\rangle = \lambda|\psi\rangle, \tag{20}$$

with $\lambda \in \mathbb{T}$ and $|\psi\rangle \in M$ such that $|\psi\rangle \notin \mathcal{NC}$. Then λ is called an *eigenvalue* of A and $|\psi\rangle$ is called an *eigenket* of A.

- Just as eigenvectors are normally restricted to nonzero vectors, we have restricted eigenkets to kets that are not in the null cone.

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We will used the following definition of a **bicomplex scalar product** where the physicists' convention is used.

Definition

Let M be a finite-dimensional free $\mathbb{T}$-module. Suppose that with each pair $|\psi\rangle$ and $|\phi\rangle$ in M, taken in this order, we associate a bicomplex number $(|\psi\rangle, |\phi\rangle)$ which, $\forall |\chi\rangle \in M$ and $\forall \alpha \in \mathbb{T}$, satisfies

1. $(|\psi\rangle, |\phi\rangle + |\chi\rangle) = (|\psi\rangle, |\phi\rangle) + (|\psi\rangle, |\chi\rangle)$;
2. $(|\psi\rangle, \alpha|\phi\rangle) = \alpha(|\psi\rangle, |\phi\rangle)$;
3. $(|\psi\rangle, |\phi\rangle) = (|\phi\rangle, |\psi\rangle)^{\dagger_3}$;
4. $(|\psi\rangle, |\psi\rangle) = 0$ if and only if $|\psi\rangle = 0$.

Then we say that $(|\psi\rangle, |\phi\rangle)$ is a *bicomplex scalar product*.

Property 3 implies that $(|\psi\rangle, |\psi\rangle) \in \mathbb{D}$. Definition 2 is very general. In this paper we shall be a little more restrictive, by requiring the bicomplex scalar product to be hyperbolic positive, that is,

$$(|\psi\rangle, |\psi\rangle) \in \mathbb{D}^+, \qquad \forall |\psi\rangle \in M. \tag{21}$$

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Let $\{|m_l\rangle\}$ be a basis of M and let V be the associated vector space. We say that a scalar product is $\mathbb{C}(\mathbf{i_1})$-*closed* under V if, $\forall|\psi\rangle, |\phi\rangle \in V$, we have $(|\psi\rangle, |\phi\rangle) \in \mathbb{C}(\mathbf{i_1})$.

We note that the property of being $\mathbb{C}(\mathbf{i_1})$-closed is basis-dependent. That is, a scalar product may be $\mathbb{C}(\mathbf{i_1})$-closed under V defined through a basis $\{|m_l\rangle\}$, but not under V' defined through a basis $\{|s_l\rangle\}$. However, one can show that for $k = 1, 2$, the following projection of a bicomplex scalar product:

$$(\cdot,\cdot)_{\widehat{k}} := P_k((\cdot,\cdot)) : M \times M \longrightarrow \mathbb{C}(\mathbf{i_1}) \tag{22}$$

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Now, we present this following important decomposition.

Theorem

Let $|\psi\rangle, |\phi\rangle \in M$, then

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Theorem 4 is true whether the bicomplex scalar product is $\mathbb{C}(\mathbf{i_1})$-closed under V or not. When it is $\mathbb{C}(\mathbf{i_1})$-closed, we have for $k = 1, 2$

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- Since $(\cdot,\cdot)_{\widehat{k}}$ is a standard scalar product when the vector space M' is restricted to V or V_k, then $(V,(\cdot,\cdot)_{\widehat{k}})$ and $(V_k,(\cdot,\cdot)_{\widehat{k}})$ are complex $(\mathbb{C}(\mathbf{i_1}))$ pre-Hilbert spaces. Hence, $(V_k,(\cdot,\cdot)_{\widehat{k}})$ are finite-dimensional normed spaces over $\mathbb{C}(\mathbf{i_1})$. Therefore they are also complete metric spaces. Hence V and V_k are complex $(\mathbb{C}(\mathbf{i_1}))$ Hilbert spaces.
- Let $|\psi_k\rangle$ and $|\phi_k\rangle$ be in V_k for $k=1,2$. On the direct sum of the two Hilbert spaces V_1 and V_2, one can define a scalar product as follows:

$$(|\psi_1\rangle \oplus |\psi_2\rangle, |\phi_1\rangle \oplus |\phi_2\rangle) = (|\psi_1\rangle, |\phi_1\rangle)_{\widehat{1}} + (|\psi_2\rangle, |\phi_2\rangle)_{\widehat{2}}. \qquad (25)$$

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From a set-theoretical point of view, M and M' are identical. In this sense we can say, perhaps improperly, that the **module** M can be decomposed into the direct sum of two classical Hilbert spaces, i.e. $M = V_1 \oplus V_2$. Now let us consider the following **norm** on the vector space M':

$$\big|\big||\phi\rangle\big|\big| := \frac{1}{\sqrt{2}}\sqrt{\big(|\phi\rangle_{\widehat{1}}, |\phi\rangle_{\widehat{1}}\big)_{\widehat{1}} + \big(|\phi\rangle_{\widehat{2}}, |\phi\rangle_{\widehat{2}}\big)_{\widehat{2}}}\,. \tag{26}$$

Making use of this norm, we can define a metric on M:

$$d(|\phi\rangle, |\psi\rangle) = \big|\big||\phi\rangle - |\psi\rangle\big|\big|. \tag{27}$$

With this metric M is **complete**, and therefore a **bicomplex Hilbert space**. We note that a bicomplex scalar product is **completely characterized** by the two scalar products $(\cdot,\cdot)_{\widehat{k}}$ on V. In fact, if $(\cdot,\cdot)_{\widehat{1}}$ and $(\cdot,\cdot)_{\widehat{2}}$ are two arbitrary scalar products on V, then $(\cdot,\cdot)$ defined in (23) is a bicomplex scalar product on M.

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- The fact that a basis of V can always be orthonormalized have the interesting consequence that it is always possible to construct an orthonormalized basis in M when the bicomplex scalar procduct is $\mathbb{C}(\mathbf{i_1})$-closed under V. Indeed, let $\{|u_i\rangle\}$ be a orthonormalized basis of V and let us construct the kets $\{|v_i\rangle\}$ as $|v_i\rangle := \mathbf{e_1}|u_i\rangle + \mathbf{e_2}|u_i\rangle$ for $i \in \{1, \ldots, n\}$. Then, we have

$$(|v_i\rangle, |v_j\rangle) = \mathbf{e_1}\,(|u_i\rangle, |u_j\rangle) + \mathbf{e_2}\,(|u_i\rangle, |u_j\rangle) = \delta_{ij}(\mathbf{e_1} + \mathbf{e_2}) = \delta_{ij}. \quad (28)$$

- Thus, $\{|v_i\rangle\}$ is an orthonomalized basis of M. Unfortunately, the same specific process cannot be used if we take another basis of M because the bicomplex scalar product will not be automatically $\mathbb{C}(\mathbf{i_1})$-closed under the new vector space V' associated with a new basis on M. However, using the representation (23) with V_1 and V_2, it is always possible to construct an orthonormalized basis in M. In fact, let M be a finite-dimensional free $\mathbb{T}$-module and let $\{|s_l\rangle\}$ be an arbitrary basis of M. Then one can always find bicomplex linear combinations of the $|s_l\rangle$ which make up an orthonormal basis.

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It is interesting to noted that the normalizability of a ket required the property $(|\psi\rangle, |\psi\rangle) \in \mathbb{D}^+$, $\forall |\psi\rangle \in M$. Let us write $(|m_1\rangle, |m_1\rangle) = a_{\hat{1}}\mathbf{e_1} + a_{\hat{2}}\mathbf{e_2}$ with $a_{\hat{1}}, a_{\hat{2}} \in \mathbb{R}$, and let

$$|m_1'\rangle = (z_{\hat{1}}\mathbf{e_1} + z_{\hat{2}}\mathbf{e_2})|m_1\rangle.$$

with $z_{\hat{1}}, z_{\hat{2}} \in \mathbb{C}(\mathbf{i_1})$ and $z_{\hat{1}} \neq 0 \neq z_{\hat{2}}$. From this, we have

$$\begin{aligned}(|m_1'\rangle, |m_1'\rangle) &= (|z_{\hat{1}}|^2\mathbf{e_1} + |z_{\hat{2}}|^2\mathbf{e_2})\,(|m_1\rangle, |m_1\rangle)\\ &= (|z_{\hat{1}}|^2\mathbf{e_1} + |z_{\hat{2}}|^2\mathbf{e_2})(a_{\hat{1}}\mathbf{e_1} + a_{\hat{2}}\mathbf{e_2})\\ &= c_{\hat{1}}a_{\hat{1}}\mathbf{e_1} + c_{\hat{2}}a_{\hat{2}}\mathbf{e_2},\end{aligned}$$

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Theorem

Let $f : M \to \mathbb{T}$ be a linear functional on M. Then there is a unique $|\psi\rangle \in M$ such that $\forall |\phi\rangle$, $f(|\phi\rangle) = (|\psi\rangle, |\phi\rangle)$.

Proof.

We make use of the analogue theorem on V, with the functional $P_k(f)$ restricted to V. The theorem shows that for each $k = 1, 2$, there is a unique $|\psi_k\rangle \in V$ such that

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Making use of the decomposition of the bicomplex scalar product, we find that $|\psi\rangle := \mathbf{e_1}|\psi_1\rangle + \mathbf{e_2}|\psi_2\rangle$ has the desired properties. □

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In the last Theorem, we showed that with finite-dimensional free $\mathbb{T}$-modules, linear functionals are in one-to-one correspondence with kets and act like scalar products. This allows for the introduction of Dirac's bra notation and the alternative writing of the scalar product $(|\psi\rangle, |\phi\rangle)$ as $\langle\psi|\phi\rangle$.

The bicomplex *adjoint* operator A^* of A is the unique operator that satisfies

$$(|\psi\rangle, A|\phi\rangle) = (A^*|\psi\rangle, |\phi\rangle), \qquad \forall|\psi\rangle, |\phi\rangle \in M. \tag{29}$$

In finite-dimensional free $\mathbb{T}$-modules the adjoint always exists, is linear and satisfies

$$(A^*)^* = A, \quad (sA + tB)^* = s^{\dagger_3}A^* + t^{\dagger_3}B^*, \quad (AB)^* = B^*A^*. \tag{30}$$

Moreover,

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Let $|\psi\rangle, |\phi\rangle \in M$. Define an operator $|\phi\rangle\langle\psi|$ so that its action on an arbitrary ket $|\chi\rangle$ is given by $(|\psi\rangle\langle\phi|)|\chi\rangle = |\phi\rangle\,(\langle\psi|\chi\rangle)$. Then $|\phi\rangle\langle\psi|$ is a linear operator on M.

Theorem

Let $\{|u_l\rangle\}$ be an orthonormal basis of M. Then

$$\sum_{l=1}^{n} |u_l\rangle\langle u_l| = I.$$

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A bicomplex linear operator H is called *self-adjoint* if $H^* = H$.

Lemma

Let $H : M \to M$ be a self-adjoint operator. Then $P_k(H) : V \to V$ $(k = 1, 2)$ is a self-adjoint operator on V.

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With the structure we have now built, we can prove the spectral decomposition theorem for finite-dimensional bicomplex Hilbert spaces.

Theorem [Bicomplex Spectral Theorem]

Let M be a finite-dimensional free $\mathbb{T}$-module and let $H : M \to M$ be a bicomplex self-adjoint operator. It is always possible to find a set $\{|\phi_l\rangle\}$ of eigenkets of H that make up an orthonormalized basis of M. Moreover, H can be expressed as

$$H = \sum_{l=1}^{n} \lambda_l |\phi_l\rangle\langle\phi_l|, \tag{32}$$

where λ_l is the eigenvalue of H associated with the eigenket $|\phi_l\rangle$.

Proof.

We first remark that the classical spectral decomposition theorem holds for the self-adjoint operator $P_k(H) = H_{\widehat{k}}$, restricted to V $(k = 1, 2)$. So let $\{|\phi_l\rangle_{\widehat{1}}\}$ and $\{|\phi_l\rangle_{\widehat{2}}\}$ be orthonormal sets of eigenvectors of $H_{\widehat{1}}$ and $H_{\widehat{2}}$, respectively. They make up orthonormal bases of V with respect to the scalar products $(\cdot,\cdot)_{\widehat{1}}$ and $(\cdot,\cdot)_{\widehat{2}}$. Letting $|\phi_l\rangle := \mathbf{e_1}|\phi_l\rangle_{\widehat{1}} + \mathbf{e_2}|\phi_l\rangle_{\widehat{2}}$, we can see that $\{|\phi_l\rangle\}$ is an orthonormal basis of M. Let λ_l be the eigenvalue of H associated with $|\phi_l\rangle$, so that $H|\phi_l\rangle = \lambda_l|\phi_l\rangle$. To show that (32) holds, it is enough to show that the right-hand side of (32) acts on basis kets like H. But

$$\left[\sum_{l=1}^{n} \lambda_l |\phi_l\rangle\langle\phi_l|\right] |\phi_p\rangle = \sum_{l=1}^{n} \lambda_l |\phi_l\rangle(\langle\phi_l|\phi_p\rangle) = \sum_{l=1}^{n} \lambda_l \delta_{lp} |\phi_l\rangle = \lambda_p |\phi_p\rangle.$$

□

Applications

As an application of the results obtained in the previous sections, we will develop the bicomplex version of the quantum-mechanical evolution operator. To do this, we first need to define bicomplex unitary operators as well as functions of a bicomplex operator.

Definition

A bicomplex linear operator U is called *unitary* if $U^*U = I$.

From the last definition one easily sees that the action of a bicomplex unitary operator preserves scalar products. Indeed let $|\psi\rangle, |\phi\rangle \in M$ and let U be unitary. Then

$$(U|\psi\rangle, U|\phi\rangle) = (U^*U|\psi\rangle, |\phi\rangle) = (I|\psi\rangle, |\phi\rangle) = (|\psi\rangle, |\phi\rangle). \qquad (33)$$

Lemma

Let $U : M \to M$ be a unitary operator. Then $P_k(U) : V \to V$ $(k = 1, 2)$ is a unitary operator on V.

We note that a bicomplex unitary operator cannot be in the null cone. For if it were, its determinant would also be in the null cone and the operator would not have an inverse.

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Lemma

Any eigenvalue λ of a bicomplex unitary operator satisfies $\lambda^{\dagger_3}\lambda = 1$.

Corollary

Let U be a unitary operator and let $|\phi\rangle \in M$ be an eigenket of U associated with the eigenvalue λ. Then $U^*|\phi\rangle = \lambda^{\dagger_3}|\phi\rangle$.

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Let M be a finite-dimensional free $\mathbb{T}$-module and let A be a linear operator acting on M. Let $A^0 := I$ and let $\{c_n \mid n = 0, 1, \ldots\}$ be an infinite sequence of bicomplex numbers. Formally we can write the infinite sum

$$\sum_{n=0}^{\infty} c_n A^n. \tag{34}$$

When this series converges to an operator acting on M, we call this operator $f(A)$.

The operator A and the coefficients c_n can be written in the idempotent basis as

$$A = \mathbf{e_1} A_{\widehat{1}} + \mathbf{e_2} A_{\widehat{2}}, \qquad c_n = \mathbf{e_1} c_{n\widehat{1}} + \mathbf{e_2} c_{n\widehat{2}}. \tag{35}$$

Substituting (35) into (34), we get

$$\begin{aligned} f(A) = \sum_{n=0}^{\infty} c_n A^n &= \mathbf{e_1} \sum_{n=0}^{\infty} c_{n\widehat{1}} A_{\widehat{1}}^n + \mathbf{e_2} \sum_{n=0}^{\infty} c_{n\widehat{2}} A_{\widehat{2}}^n \\ &= \mathbf{e_1} f_1(A_{\widehat{1}}) + \mathbf{e_2} f_2(A_{\widehat{2}}). \end{aligned} \tag{36}$$

One can see that the f series converges if and only if the two series f_1 and f_2 converge.

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A very important bicomplex function of an operator is of course the *exponential*, defined in the usual way as

$$\exp\{A\} = I + \sum_{n=1}^{\infty} \frac{1}{n!} A^n. \tag{37}$$

Clearly,

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We have these two important theorems on exponentials of operators.

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If t is a real parameter, $\frac{d}{dt}\exp\{tA\} = A\exp\{tA\}$.

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A generalization of the Schrödinger equation was already proposed for bicomplex numbers. It can be adapted to finite-dimensional modules as

$$\mathbf{i_1}\hbar\frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle. \tag{39}$$

From the last Theorems we immediately obtain

Theorem

If H doesn't depend on time, solutions of (39) are given by $|\psi(t)\rangle = U(t,t_0)|\psi(t_0)\rangle$, where $|\psi(t_0)\rangle$ is any ket and

$$U(t,t_0) = \exp\left\{-\frac{\mathbf{i_1}}{\hbar}(t-t_0)H\right\}.$$

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CONCLUSION

We have derived a number of new results on finite-dimensional bicomplex matrices, modules, operators and Hilbert spaces, including the generalization of the spectral decomposition theorem. All these concepts are deeply connected with the formalism of quantum mechanics. We believe that many if not all of them can be extended to infinite-dimensional Hilbert spaces.