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Bicomplex Hilbert Space

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Bicomplex Numbers

We define **bicomplex numbers** as follows:

$$\mathbb{T} := \{a + b\mathbf{i_1} + c\mathbf{i_2} + d\mathbf{j} : \mathbf{i_1}^2 = \mathbf{i_2}^2 = -1,\ \mathbf{j}^2 = 1\}.$$

Where

$$\begin{aligned} \mathbf{i_2}\mathbf{j} &= \mathbf{j}\mathbf{i_2} = -\mathbf{i_1} \\ \mathbf{i_1}\mathbf{j} &= \mathbf{j}\mathbf{i_1} = -\mathbf{i_2} \\ \mathbf{i_2}\mathbf{i_1} &= \mathbf{i_1}\mathbf{i_2} = \mathbf{j} \end{aligned}$$

and $a, b, c, d \in \mathbb{R}$.

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Bicomplex Numbers

We remark that we can write a bicomplex number

$$a + b\mathbf{i_1} + c\mathbf{i_2} + d\mathbf{j}$$

as

$$(a + b\mathbf{i_1}) + (c + d\mathbf{i_1})\mathbf{i_2} = z_1 + z_2\mathbf{i_2}$$

where

$$z_1, z_2 \in \mathbb{C}(\mathbf{i_1}) := \{x + y\mathbf{i_1} : \mathbf{i_1}^2 = -1\}.$$

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Bicomplex Numbers

It is also important to know that every bicomplex number $z_1 + z_2\mathbf{i_2}$ has the following unique idempotent representation:

$$z_1 + z_2\mathbf{i_2} = (z_1 - z_2\mathbf{i_1})\mathbf{e_1} + (z_1 + z_2\mathbf{i_1})\mathbf{e_2}$$

where $\mathbf{e_1} = \frac{1+\mathbf{j}}{2}$ and $\mathbf{e_2} = \frac{1-\mathbf{j}}{2}$. This representation is very useful because: addition, multiplication and division can be done term-by-term.

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Bicomplex Numbers

Complex conjugation plays an important role both for algebraic and geometric properties of $\mathbb{C}$, as well as in the standard quantum mechanics. For bicomplex numbers, there are three possible conjugations. Let $w \in \mathbb{T}$ and $z_1, z_2 \in \mathbb{C}(\mathbf{i_1})$ such that $w = z_1 + z_2\mathbf{i_2}$. Then we define the three conjugations as:

(1) $w^{\dagger_1} = (z_1 + z_2\mathbf{i_2})^{\dagger_1} := \overline{z}_1 + \overline{z}_2\mathbf{i_2}$;

(2) $w^{\dagger_2} = (z_1 + z_2\mathbf{i_2})^{\dagger_2} := z_1 - z_2\mathbf{i_2}$;

(3) $w^{\dagger_3} = (z_1 + z_2\mathbf{i_2})^{\dagger_3} := \overline{z}_1 - \overline{z}_2\mathbf{i_2}$.

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Bicomplex Numbers

The bicomplex moduli are defined as follow:

$$|w|^2_{\mathbf{i_1}} := w \cdot w^{\dagger_2} = z_1^2 + z_2^2 \in \mathbb{C}(\mathbf{i_1}),$$

$$|w|^2_{\mathbf{i_2}} := w \cdot w^{\dagger_1} = \left(|z_1|^2 - |z_2|^2\right) + 2\mathrm{Re}(z_1\overline{z}_2)\mathbf{i_2} \in \mathbb{C}(\mathbf{i_2}),$$

$$|w|^2_{\mathbf{j}} := w \cdot w^{\dagger_3} = \left(|z_1|^2 + |z_2|^2\right) - 2\mathrm{Im}(z_1\overline{z}_2)\mathbf{j} \in \mathbb{D}.$$

It is easy to verify that the inverse of w is given by

$$w^{-1} = \frac{w^{\dagger_2}}{|w|^2_{\mathbf{i_1}}}.$$

The set $\mathcal{NC}$ of zero divisors of $\mathbb{T}$, called the *null-cone*, is given by

$$\mathcal{NC} = \{z(\mathbf{i_1} \pm \mathbf{i_2}) |\ z \in \mathbb{C}(\mathbf{i_1})\}.$$

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Bicomplex Numbers

Let $s, t \in \mathbb{T}$, we define the *real* moduli $|\cdot|_k$, $k = 1, 2, 3$ as

1. $|\cdot|_{\mathbf{1}} := \big||\cdot|_{\mathbf{i_1}}\big|$,

   For $w = z_1 + z_2\mathbf{i_2}$ with $z_1, z_2 \in \mathbb{C}(\mathbf{i_1})$ we have

   $$|w|_{\mathbf{1}} = |z_1^2 + z_2^2|^{1/2} = \sqrt[4]{ww^{\dagger_1}w^{\dagger_2}w^{\dagger_3}}.$$

   This modulus has the following properties:

   (a) $|\cdot|_{\mathbf{1}} : \mathbb{T} \to \mathbb{R}$;

   (b) $|s|_{\mathbf{1}} \geq 0$ with $|s|_{\mathbf{1}} = 0$ iff $s \in \mathcal{NC}$;

   (c) $|s \cdot t|_{\mathbf{1}} = |s|_{\mathbf{1}} \cdot |t|_{\mathbf{1}}$.

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Bicomplex Numbers

2. $|\cdot|_{\mathbf{2}} := \big||\cdot|_{\mathbf{i_2}}\big|$,

   This modulus has the same properties as $|\cdot|_{\mathbf{1}}$. In fact,

$$|w|_{\mathbf{2}} = |w|_{\mathbf{1}} = \sqrt[4]{ww^{\dagger_1}w^{\dagger_2}w^{\dagger_3}}.$$

3. $|\cdot|_{\mathbf{3}} := \big||\cdot|_{\mathbf{j}}\big|$,

   For $w = z_1 + z_2\mathbf{i_2}$ with $z_1, z_2 \in \mathbb{C}(\mathbf{i_1})$ we have

$$|w|_{\mathbf{3}} = |w| = \sqrt{|z_1|^2 + |z_2|^2}.$$

   This modulus has the following properties:

   (a) $|\cdot|_{\mathbf{3}} : \mathbb{T} \to \mathbb{R}$ (b) $|s|_{\mathbf{3}} \geq 0$ with $|s|_{\mathbf{3}} = 0$ iff $s = 0$

   (c) $|s+t|_{\mathbf{3}} \leq |s|_{\mathbf{3}} + |t|_{\mathbf{3}}$ (d) $|s \cdot t|_{\mathbf{3}} \leq \sqrt{2}\ |s|_{\mathbf{3}} \cdot |t|_{\mathbf{3}}$.

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$\mathbb{T}$-Module

The set of bicomplex number is a commutative ring. So, to define a kind of vector space over $\mathbb{T}$, we have to deal with the algebraic concept of module. We denote M as a free $\mathbb{T}$-module with the following finit $\mathbb{T}$-basis $\Big\{\widehat{m}_l \mid l \in \{1,\ldots,n\}\Big\}$. Hence,

$$M = \left\{\sum_{l=1}^{n} x_l\widehat{m}_l \mid x_l \in \mathbb{T}\right\}.$$

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$\mathbb{T}$-Module

Let us now define

$$V := \left\{ \sum_{l=1}^{n} x_l \widehat{m}_l \mid x_l \in \mathbb{C}(\mathbf{i_1}) \right\} \subset M.$$

The set V is a free $\mathbb{C}(\mathbf{i_1})$-module which depends on a given $\mathbb{T}$-basis of M. In fact, V is a complex vector space of dimension n with the basis $\left\{ \widehat{m}_l \mid l \in \{1, \ldots, n\} \right\}$.

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Bicomplex scalar product

Let us begin with a preliminary definition.

The set

$$\mathbb{D} := \{x + y\mathbf{j} | x, y \in \mathbb{R}\}$$

will be called the set of hyperbolic numbers (also called duplex numbers).

Definition 1 *A hyperbolic number* $w = a\mathbf{e_1} + b\mathbf{e_2}$ *will be defined positive if* $a, b \in \mathbb{R}^+$. *We denote the set of all positive hyperbolic numbers by*

$$\mathbb{D}^+ := \{a\mathbf{e_1} + b\mathbf{e_2} \mid a, b \geq 0\}.$$

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Bicomplex scalar product

We are now able to give a definition of a bicomplex scalar product.

Definition 2 *Let M be a free $\mathbb{T}$-module of finit dimension. With each pair $\widehat{X}$ and $\widehat{Y}$ in M, taken in this order, we associate a bicomplex number, which is their bicomplex scalar product $(\widehat{X}, \widehat{Y})$, and which satisfies the following properties:*

1. $(\widehat{X}, \widehat{Y}_1 + \widehat{Y}_2) = (\widehat{X}, \widehat{Y}_1) + (\widehat{X}, \widehat{Y}_2)$, $\forall \widehat{X}, \widehat{Y}_1, \widehat{Y}_2 \in M$;
2. $(\widehat{X}, \alpha\widehat{Y}) = \alpha(\widehat{X}, \widehat{Y})$, $\forall \alpha \in \mathbb{T}$, $\forall \widehat{X}, \widehat{Y} \in M$;
3. $(\widehat{X}, \widehat{Y}) = (\widehat{Y}, \widehat{X})^{\dagger_3}$, $\forall \widehat{X}, \widehat{Y} \in M$;
4. $(\widehat{X}, \widehat{X}) = 0 \;\Leftrightarrow\; \widehat{X} = 0$, $\forall \widehat{X} \in M$.

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Bicomplex scalar product

As a consequence of property 3, we have that $(\widehat{X}, \widehat{X}) \in \mathbb{D}$. Note that definition 2 is a general definition of a bicomplex scalar product. However, in this talk we will also suppose the bicomplex scalar product $(\cdot, \cdot)$ to be *hyperbolic positive*, i.e.

$$(\widehat{X}, \widehat{X}) \in \mathbb{D}^+, \;\; \forall \widehat{X} \in M$$

and *closed* under the vector space V, i.e.

$$(\widehat{X}, \widehat{Y}) \in \mathbb{C}(\mathbf{i_1}), \;\; \forall \widehat{X}, \widehat{Y} \in V.$$

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Complementary Projections

Whenever $\widehat{X} = \sum_{l=1}^{n} x_l \widehat{m}_l \in M$, $x_l = x_{1l}\mathbf{e_1} + x_{2l}\mathbf{e_2}$ where $x_{1l}, x_{2l} \in \mathbb{C}(\mathbf{i_1})$, for all $l \in \{1, \ldots, n\}$, we define the projection $P_k : M \longrightarrow V$ as

$$P_k(\widehat{X}) := \widehat{X}_{\mathbf{e_k}} = \sum_{l=1}^{n} (x_{kl} \widehat{m}_l)$$

for $k = 1, 2$.

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Complementary Projections

This definition is a generalization of the mutually complementary projections $\{P_1, P_2\}$ defined on $\mathbb{T}$, where $\mathbb{T}$ is considered as the canonical $\mathbb{T}$-module over the ring of bicomplex numbers. Moreover, $\widehat{X}_{\mathbf{e_1}}$ and $\widehat{X}_{\mathbf{e_2}}$ are uniquely determined from a given $\mathbb{T}$-basis and the projections P_1 and P_2 satisfies the following property:

$$P_k(w_1\widehat{X} + w_2\widehat{Y}) = P_k(w_1)P_k(\widehat{X}) + P_k(w_2)P_k(\widehat{Y})$$

$\forall w_1, w_2 \in \mathbb{T}$, $\forall \widehat{X}, \widehat{Y} \in M$ and $k = 1, 2$.

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Norm over V

Theorem 1 *$\{V;(\cdot,\cdot)\}$ is a complex (in $\mathbb{C}(\mathbf{i_1})$) pre-Hilbert space.*

Let us denote $\| \widehat{X} \| := (\widehat{X},\widehat{X})^{\frac{1}{2}}, \forall \widehat{X} \in V$.

Corollary 1 *Let $\widehat{X} \in V$. The function $\widehat{X} \longmapsto \| \widehat{X} \| \geq 0$ is a norm on V.*

Theorem 2 *Let $\widehat{X} \in M$, then*

$$P_k\big((\widehat{X},\widehat{X})\big) = (\widehat{X},\widehat{X})_{\mathbf{e_k}} = (\widehat{X}_{\mathbf{e_k}},\widehat{X}_{\mathbf{e_k}}) = \| \widehat{X}_{\mathbf{e_k}} \|^2$$

for $k = 1, 2$.

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Norm over M

Now, let us extend this norm on M with the following function:

$$\| \widehat{X} \| := \left|(\widehat{X},\widehat{X})^{\frac{1}{2}}\right| = \Big|\mathbf{e_1} \| \widehat{X}_{\mathbf{e_1}} \| + \mathbf{e_2} \| \widehat{X}_{\mathbf{e_2}} \| \Big|, \ \forall \widehat{X} \in M.$$

This *norm* has the following properties.

Theorem 3 *Let $\widehat{X},\widehat{Y} \in M$ and $d(\widehat{X},\widehat{Y}) := \| \widehat{X} - \widehat{Y} \|$, then*
1. $\| \widehat{X} \| \geq 0$*;*
2. $\| \widehat{X} \| = 0 \ \Leftrightarrow \ \widehat{X} = 0$*;*
3. $\| \alpha\widehat{X} \| = |\alpha| \, \| \widehat{X} \|, \forall \alpha \in \mathbb{C}(\mathbf{i_1})$ *or* $\mathbb{C}(\mathbf{i_2})$*;*
4. $\| \alpha\widehat{X} \| \leq \sqrt{2}\, |\alpha|_{\mathbf{3}} \, \| \widehat{X} \|, \forall \alpha \in \mathbb{T}$*;*
5. $\| \widehat{X} + \widehat{Y} \| \leq \| \widehat{X} \| + \| \widehat{Y} \|$*;*
6. $\{M, d\}$ *is a* ***metric space.***

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Bicomplex Hilbert space

Definition 3 *Let M be a free $\mathbb{T}$-module with a finit $\mathbb{T}$-basis. Let also $(\cdot,\cdot)$ be a bicomplex scalar product defined on M. The space $\{M,(\cdot,\cdot)\}$ is called a $\mathbb{T}$-inner product space.*

Definition 4 *A complete $\mathbb{T}$-inner product space is called a $\mathbb{T}$-Hilbert space.*

Theorem 4 *A $\mathbb{T}$-inner product space $\{M,(\cdot,\cdot)\}$ is a $\mathbb{T}$-Hilbert space if and only if $\{V,(\cdot,\cdot)\}$ is an Hilbert space.*

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Example of bicomplex Hilbert space

Let us consider $M = \mathbb{T}$, the canonical $\mathbb{T}$-module over the ring of bicomplex numbers. We consider now the trivial $\mathbb{T}$-basis $\{1\}$. In this case, the submodule vector space V is simply $V = \mathbb{C}(\mathbf{i_1})$.

Consider now $\{\mathbb{C}(\mathbf{i_1}),(\cdot,\cdot)\}$ with the canonical scalar product given by

$$\begin{aligned}(z_1,z_2) &= (x_1 + y_1\mathbf{i_1}, x_2 + y_2\mathbf{i_1}) \\ &:= x_1x_2 + y_1y_2.\end{aligned}$$

It is well known that $\{\mathbb{C}(\mathbf{i_1}),(\cdot,\cdot)\}$ is an Hilbert space. So, from the last Theorem, $\{\mathbb{T},(\cdot,\cdot) := (\cdot,\cdot)\mathbf{e_1} + (\cdot,\cdot)\mathbf{e_2}\}$ is a bicomplex Hilbert space. Moreover, it is easy to see that

$$\| w \| = ||w|_{\mathbf{j}}| = |w|_{\mathbf{3}} = |w|,$$

i.e. the Euclidean metric of $\mathbb{R}^4$.

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The Dirac notation over M

In this section we introduce the Dirac notation usually used in quantum mechanics. For this we have to define correctly kets and bras over a bicomplex Hilbert space which, we remind, is fundamentally a module.

Let M be a free $\mathbb{T}$-module with the following finit $\mathbb{T}$-basis $\{|m_l\rangle \mid l \in \{1, \ldots, n\}\}$. Any element of M will be called a *ket module* or, more simply, a *ket*.

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The Dirac notation over M

Definition 5 *A linear functional χ is a linear operation which associates a bicomplex number with every ket $|\psi\rangle$:*

1) $|\psi\rangle \longrightarrow \chi(|\psi\rangle) \in \mathbb{T}$;

2) $\chi(\lambda_1|\psi_1\rangle + \lambda_2|\psi_2\rangle) = \lambda_1\chi(|\psi_1\rangle) + \lambda_2\chi(|\psi_2\rangle), \quad \lambda_1, \lambda_2 \in \mathbb{T}$.

It can be shown that the set of linear functionals defined on the kets $|\psi\rangle \in M$ constitutes a $\mathbb{T}$-module space, which is called the dual space of M and which will be symbolized by M^*.

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The Dirac notation over M

Using this definition of M^*, let us define the bra notation.

Definition 6 *Any element of the space M^* is called a bra module or, more simply, a bra. It is symbolized by $\langle \cdot |$.*

For example, the bra $\langle \chi|$ designates the bicomplex linear functional χ and we shall henceforth use the notation $\langle \chi|\psi\rangle$ to denote the number obtained by causing the linear functional $\langle \chi| \in M^*$ to act on the ket $|\psi\rangle \in M$:

$$\chi(|\psi\rangle) := \langle \chi|\psi\rangle.$$

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The Dirac notation over M

The existence of a bicomplex scalar product in M will now enable us to show that we can associate, with every ket $|\phi\rangle \in M$, an element of M^*, which will be denoted by $\langle \phi|$.

The ket $|\phi\rangle$ does indeed enable us to define a linear functional: the one which associates (in a linear way), with each ket $|\psi\rangle \in M$, a bicomplex numbers which is equal to the scalar product $(|\phi\rangle, |\psi\rangle)$ of $|\psi\rangle$ by $|\phi\rangle$. Let $\langle \phi|$ be this linear functional; It is thus defined by the relation:

$$\langle \phi|\psi\rangle = (|\phi\rangle, |\psi\rangle).$$

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The Dirac notation over M

Therefore, the properties of the bicomplex scalar product can be rewrited as:

1. $\langle\phi|\big(|\psi_1\rangle + |\psi_2\rangle\big) = \langle\phi|\psi_1\rangle + \langle\phi|\psi_2\rangle$;
2. $\langle\phi|\alpha\psi\rangle = \alpha\,\langle\phi|\psi\rangle$, $\forall\alpha\in\mathbb{T}$;
3. $\langle\phi|\psi\rangle = \langle\psi|\phi\rangle^{\dagger_3}$;
4. $\langle\phi|\phi\rangle = 0 \;\Leftrightarrow\; |\phi\rangle = 0$.

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The Dirac notation over M

Now, let define the corresponding projections for the Dirac notation as follows.

Definition 7 *Let $|\psi\rangle$,$|\phi\rangle \in M$ and $|\chi\rangle \in V$. For $k = 1, 2$, we define:*

1. $|\psi_{\mathbf{e_k}}\rangle := P_k(|\psi\rangle) \in V$;
2. $\langle\phi_{\mathbf{e_k}}| := P_k(\langle\phi|) : V \longrightarrow \mathbb{C}(\mathbf{i_1})$, *where* $|\chi\rangle \mapsto P_k\big(\langle\phi|\chi\rangle\big)$.

The first definition gives the projection $|\psi_{\mathbf{e_k}}\rangle$ of the ket $|\psi\rangle$ of M. The second definition is more subtle. In the next two theorems, we show that the functional $\langle\phi_{\mathbf{e_k}}|$ is really the bra associated with the ket $|\phi_{\mathbf{e_k}}\rangle$ in V.

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The Dirac notation over M

Theorem 5 *Let* $|\phi\rangle \in M$*, then*

$$\langle \phi_{\mathbf{e_k}}| \in V^*$$

for $k = 1, 2$*.*

Theorem 6 *Let* $|\phi\rangle \in M$ *and* $|\psi\rangle \in V$*, then*

$$\langle \phi_{\mathbf{e_k}}|(|\psi\rangle) = \langle \phi_{\mathbf{e_k}}|\psi\rangle$$

for $k = 1, 2$*.*

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Bicomplex linear operators

The *bicomplex linear operators* $A : M \to M$ are defined by

$$|\psi'\rangle = A|\psi\rangle,$$

$$A(\lambda_1|\psi_1\rangle + \lambda_2|\psi_2\rangle) = \lambda_1 A|\psi_1\rangle + \lambda_2 A|\psi_2\rangle,$$

where $\lambda_1, \lambda_2 \in \mathbb{T}$. For a fixed $|\phi\rangle \in M$, a fixed linear operator A and an arbitrary $|\psi\rangle \in M$, we define the bra $\langle\phi|A$ by the relation

$$\big(\langle\phi|A\big)|\psi\rangle := \langle\phi|\big(A|\psi\rangle\big).$$

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Bicomplex linear operators

For a given linear operator $A : M \rightarrow M$, the *bicomplex adjoint operator* A^* is the operator with the following correspondance

$$|\psi'\rangle = A|\psi\rangle \leftrightsquigarrow \langle\psi'| = \langle\psi|A^*.$$

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Bicomplex linear operators

It is easy to show that for any bicomplex linear operator $A : M \rightarrow M$ and $\lambda \in \mathbb{T}$, we have the following standard properties:

$$\begin{aligned}
(A^*)^* &= A, \\
(\lambda A)^* &= \lambda^{\dagger_3} A^*, \\
(A+B)^* &= A^* + B^*, \\
(AB)^* &= B^* A^*.
\end{aligned}$$

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Bicomplex linear operators

Definition 8 *Let M be a bicomplex Hilbert space and $A : M \to M$ a bicomplex linear operator. We define the projection $P_k(A) : M \to V$ of A, for $k = 1, 2$, as follows :*

$$P_k(A)|\psi\rangle := P_k(A|\psi\rangle), \quad \forall\, |\psi\rangle \in M.$$

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Bicomplex linear operators

We have the following specific results.

Theorem 7 *Let M be a bicomplex Hilbert space, $A : M \to M$ a bicomplex linear operator and $|\psi\rangle = \mathbf{e_1}|\psi_{\mathbf{e_1}}\rangle + \mathbf{e_2}|\psi_{\mathbf{e_2}}\rangle \in M$. Then*

(i) $A|\psi\rangle = \mathbf{e_1}P_1(A)|\psi_{\mathbf{e_1}}\rangle + \mathbf{e_2}P_2(A)|\psi_{\mathbf{e_2}}\rangle$;

(ii) $P_k(A)^* = P_k(A^*)$ *where $P_k(A)^*$ is the standard complex adjoint operator over $\mathbb{C}(\mathbf{i_1})$ associated with the bicomplex linear operator $P_k(A)$ restricted to the submodule vector space V for $k = 1, 2$.*

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Bicomplex self-adjoint operators

In standard quantum mechanics self-adjoint operators (Hermitian operators) play a very important role. In analogy with the standard case, a linear operator A is defined to be a *bicomplex self-adjoint operator* if and only if $A = A^*$.

Theorem 8 *Let $A : M \to M$ be a bicomplex self-adjoint operator and $|\psi\rangle \in M$ be an eigenvector of the equation $A|\psi\rangle = \lambda|\psi\rangle$, with $|\psi\rangle \notin \mathcal{NC}$. Then the eigenvalues of A are in the set of hyperbolic numbers.*