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On a Bicomplex Distance Estimation for the Tetrabrot

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Bicomplex Numbers

We define **bicomplex numbers** as follows:

$$\boxed{\mathbb{T} := \{a + b\mathbf{i_1} + c\mathbf{i_2} + d\mathbf{j} : \mathbf{i_1}^2 = \mathbf{i_2}^2 = -1,\ \mathbf{j}^2 = 1\}.}$$

Where

$$\begin{aligned} \mathbf{i_2}\mathbf{j} &= \mathbf{j}\mathbf{i_2} = -\mathbf{i_1} \\ \mathbf{i_1}\mathbf{j} &= \mathbf{j}\mathbf{i_1} = -\mathbf{i_2} \\ \mathbf{i_2}\mathbf{i_1} &= \mathbf{i_1}\mathbf{i_2} = \mathbf{j} \end{aligned}$$

and $a, b, c, d \in \mathbb{R}$.

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Bicomplex Numbers

We remark that we can write a bicomplex number

$$a + b\mathbf{i_1} + c\mathbf{i_2} + d\mathbf{j}$$

as

$$(a + b\mathbf{i_1}) + (c + d\mathbf{i_1})\mathbf{i_2} = z_1 + z_2\mathbf{i_2}$$

where

$$z_1, z_2 \in \mathbb{C}(\mathbf{i_1}) := \{x + y\mathbf{i_1} : \mathbf{i_1}^2 = -1\}.$$

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Bicomplex Numbers

The norm used on $\mathbb{T}$ is the Euclidean norm (noted $|\ \ |$) of $\mathbb{R}^4$ and the following formula is true over the set of bicomplex numbers:

$$|z_1 + z_2\mathbf{i_2}| = \left(\frac{|z_1 - z_2\mathbf{i_1}|^2 + |z_1 + z_2\mathbf{i_1}|^2}{2}\right)^{1/2}.$$

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Bicomplex Numbers

It is also important to know that every bicomplex number $z_1 + z_2\mathbf{i_2}$ has the following unique idempotent representation:

$$z_1 + z_2\mathbf{i_2} = (z_1 - z_2\mathbf{i_1})\mathbf{e_1} + (z_1 + z_2\mathbf{i_1})\mathbf{e_2}$$

where $\mathbf{e_1} = \frac{1+\mathbf{j}}{2}$ and $\mathbf{e_2} = \frac{1-\mathbf{j}}{2}$. This representation is very useful because: addition, multiplication and division can be done term-by-term.

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Bicomplex Numbers

Now, as a consequence of the idempotent representation, we are able to define a bicomplex cartesian product:

Definition 1 *We say that* $X \subseteq \mathbb{T}$ *is a* $\mathbb{T}$*-cartesian set determined by* X_1 *and* X_2 *if* $X = X_1 \times_e X_2 := \{z_1 + z_2\mathbf{i_2} \in \mathbb{T} : z_1 + z_2\mathbf{i_2} = w_1\mathbf{e_1} + w_2\mathbf{e_2}, (w_1, w_2) \in X_1 \times X_2\}$.

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Bicomplex Numbers

The set

$$\mathbb{D} := \{x + y\mathbf{j} | x, y \in \mathbb{R}\}$$

will be called the set of hyperbolic numbers (also called duplex numbers) and

$$|w|_{\mathbf{j}} := |z_1 - z_2\mathbf{i_1}|\mathbf{e_1} + |z_1 + z_2\mathbf{i_1}|\mathbf{e_2} \in \mathbb{D}$$

will be referred to as the *modulus in* $\mathbf{j}$ of $w = z_1 + z_2\mathbf{i_2}$

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Bicomplex Numbers

This specific modulus satisfies the following properties:

(1) $||w|_{\mathbf{j}}| = |w| = \sqrt{\mathrm{Re}(|w|_{\mathbf{j}}^2)}$;

(2) $|w|_{\mathbf{j}} = 0$ if and only if $w = 0$;

(3) $|w_1 \cdot w_2|_{\mathbf{j}} = |w_1|_{\mathbf{j}} |w_2|_{\mathbf{j}}\ \forall w_1, w_2 \in \mathbb{T}$.

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Generalized Mandelbrot Set

Now, let us define a version of the Mandelbrot set for the bicomplex numbers:

Definition 2 *Let $P_c(w) = w^2 + c$ where $w, c \in \mathbb{T}$ and $P_c^{\circ n}(w) := (P_c^{\circ(n-1)} \circ P_c)(w)$. Then the generalized Mandelbrot set for bicomplex numbers is defined as follows:*

$$\mathcal{M}_2 = \{c \in \mathbb{T} : P_c^{\circ n}(0) \nrightarrow \infty\}.$$

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Generalized Filled-Julia Sets

It is also possible to generalize the notion of filled-Julia set for the bicomplex numbers:

Definition 3 *The generalized filled-Julia set for bicomplex numbers is defined as follows:* $(c \in \mathbb{T})$

$$\mathcal{K}_{2,c} = \{w \in \mathbb{T} : P_c^{\circ n}(w) \nrightarrow \infty\}.$$

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The Tetrabrot

Because of it's rich fractal structure and his symmetry, we emphasis our work on the generalized Mandelbrot set for bicomplex numbers in dimension three:

Definition 4 *The "Tetrabrot" is defined as follows:*

$$\mathcal{T} = \{a + b\mathbf{i_1} + c\mathbf{i_2} + d\mathbf{j} \in \mathbb{T} : d = 0 \ \textit{and}\ P_c^{\circ n}(0) \nrightarrow \infty\}.$$

Distance Estimation for the Tetrabrot

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Let us begin with the following well known result about the distance estimation for the filled-Julia sets in the complex plane.

Theorem 1 *Let $d(z, \mathcal{K}_b) = \inf\{|z - a| : a \in \mathcal{K}_b\}$ be defined as the distance from $z \in \mathbb{C}$ to the filled-Julia set $\mathcal{K}_b$ with $b \in \mathcal{M}$. Then the distance $d(z_0, \mathcal{K}_b)$ between z_0 lying outside of $\mathcal{K}_b$ and $\mathcal{K}_b$ itself satisfies*

$$\frac{\sinh[G(z_0)]}{2e^{G(z_0)}|G'(z_0)|} < d(z_0, \mathcal{K}_b) < \frac{2\sinh[G(z_0)]}{|G'(z_0)|}$$

where $G(z_0)$ is the potential at the point z_0.

Distance Estimation for the Tetrabrot

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We will express the distance from a point $w \in \mathbb{T}$ to a bicomplex filled-Julia set in terms of two distances in the complex plane (in $\mathbf{i_1}$).

Lemma 1 *Let $d(w, \mathcal{K}_{2,c}) = \inf\{|w - a| : a \in \mathcal{K}_{2,c}\}$ be defined as the "bicomplex" distance from $w = z_1 + z_2\mathbf{i_2} \in \mathbb{T}$ to the bicomplex filled-Julia set $\mathcal{K}_{2,c}$ where $c = c_1 + c_2\mathbf{i_2} \in \mathbb{T}$. Hence, $d(w, \mathcal{K}_{2,c}) =$*

$$\left[\frac{[d(z_1 - z_2\mathbf{i_1}, \mathcal{K}_{c_1 - c_2\mathbf{i_1}})]^2 + [d(z_1 + z_2\mathbf{i_1}, \mathcal{K}_{c_1 + c_2\mathbf{i_1}})]^2}{2}\right]^{1/2}.$$

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Distance Estimation for the Tetrabrot

Definition 5 *Let* $G_1(z_1 - z_2\mathbf{i_1})$ *and* $G_2(z_1 + z_2\mathbf{i_1})$ *be two electrostatic potentials. The bicomplex potential, at a point* $w = z_1 + z_2\mathbf{i_2} \in (\mathbb{C}(\mathbf{i_1}) \setminus \mathcal{K}_{b_1}) \times_e (\mathbb{C}(\mathbf{i_1}) \setminus \mathcal{K}_{b_2})$*, is defined as*

$$G(w) := G_1(z_1 - z_2\mathbf{i_1})\mathbf{e_1} + G_2(z_1 + z_2\mathbf{i_1})\mathbf{e_2} \in \mathbb{D}$$

and

$$G'(w) := G_1'(z_1 - z_2\mathbf{i_1})\mathbf{e_1} + G_2'(z_1 + z_2\mathbf{i_1})\mathbf{e_2} \in \mathbb{D}.$$

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In $\mathbb{T}$, the bicomplex logarithm $\ln(z_1 + z_2\mathbf{i_2})$ is defined to be the inverse of the bicomplex exponential function $e^{z_1+z_2\mathbf{i_2}} := e^{z_1}[\cos(z_2) + \mathbf{i_2}\sin(z_2)]$. With this definition of the bicomplex logarithm, it is possible to express the bicomplex potential in a similar way to that used for one complex variable. Let $\mathbb{T} \setminus_e \mathcal{K}_{2,c} := (\mathbb{C}(\mathbf{i_1}) \setminus \mathcal{K}_{c_1 - c_2\mathbf{i_1}}) \times_e (\mathbb{C}(\mathbf{i_1}) \setminus \mathcal{K}_{c_1 + c_2\mathbf{i_1}})$.

Theorem 2 *Let* $G : \mathbb{T} \setminus_e \mathcal{K}_{2,c} \longrightarrow \mathbb{D}$ *be a bicomplex potential and* $c = (c_1 - c_2\mathbf{i_1})\mathbf{e_1} + (c_1 + c_2\mathbf{i_1})\mathbf{e_2}$*. Then,*

$$G(w) = \ln|\phi_c(w)|_{\mathbf{j}} \ \forall w \in \mathbb{T}$$

where $\phi_c : \mathbb{T} \setminus_e \mathcal{K}_{2,c} \longrightarrow \mathbb{T} \setminus_e \overline{B^1(0,1)} \times_e \overline{B^1(0,1)}$ *is biholomorphic in terms of two complex variables.*

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Distance Formulas

We are now ready to state the major result of this talk.

Theorem 3 *Let* $w_0 = z_1 + z_2\mathbf{i_2} \in \mathbb{T}$ *and* $c_1 + c_2\mathbf{i_2} \in \mathcal{M}_2$*. Then, the distance* $d(w_0, \mathcal{K}_{2,c})$ *between* w_0 *lying outside of* $\mathcal{K}_{2,c}$ *and* $\mathcal{K}_{2,c}$ *itself satisfies:*

(**1**) *If* $w_0 \in \mathbb{T} \setminus_e \mathcal{K}_{2,c}$,

$$\left| \frac{\sinh[G(w_0)]}{2e^{G(w_0)}G'(w_0)} \right| < d(w_0, \mathcal{K}_{2,c}) < \left| \frac{2\sinh[G(w_0)]}{G'(w_0)} \right|$$

where $G(w_0)$ *is the bicomplex potential at the point* w_0*.*

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(**2**) If $w_0 \in (\mathbb{C}(\mathbf{i_1}) \setminus \mathcal{K}_{c_1 - c_2\mathbf{i_1}}) \times_e (\mathcal{K}_{c_1 + c_2\mathbf{i_1}})$,

$$d(w_0, \mathcal{K}_{2,c}) > \frac{\sinh[G_1(z_1 - z_2\mathbf{i_1})]}{2\sqrt{2}e^{G_1(z_1 - z_2\mathbf{i_1})}|G_1'(z_1 - z_2\mathbf{i_1})|}$$

and

$$d(w_0, \mathcal{K}_{2,c}) < \frac{\sqrt{2}\sinh[G_1(z_1 - z_2\mathbf{i_1})]}{|G_1'(z_1 - z_2\mathbf{i_1})|}$$

(**3**) If $w_0 \in (\mathcal{K}_{c_1 - c_2\mathbf{i_1}}) \times_e (\mathbb{C}(\mathbf{i_1}) \setminus \mathcal{K}_{c_1 + c_2\mathbf{i_1}})$.

-Similar to (**2**)-

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Approximated Distance Formulas

Theorem 4 *Let $w_0 = z_1 + z_2\mathbf{i_2} \in \mathbb{T}$ and $c_1 + c_2\mathbf{i_2} \in \mathcal{M}_2$. Then, the distance $d(w_0, \mathcal{K}_{2,c})$ between w_0 lying outside of $\mathcal{K}_{2,c}$ and $\mathcal{K}_{2,c}$ itself approximatly satisfies:*

(**1**) *If $w_0 \in \mathbb{T} \setminus_e \mathcal{K}_{2,c}$,*

$$\left| \frac{w_n \ln |w_n|_{\mathbf{j}}}{2|w|_{\mathbf{j}}^{\frac{1}{2^n}} w_n'} \right| < d(w_0, \mathcal{K}_{2,c}) < \left| 2\frac{w_n}{w_n'} \ln |w_n|_{\mathbf{j}} \right|$$

where $w_n := P_c^{\circ n}(w_0)$
and $w_n' := \frac{d}{dw}[P_c^{\circ n}(w)]|_{w=w_0} \forall n \in \mathbb{N}$.

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Approximated Distance Formulas

(**2**) If $w_0 \in (\mathbb{C}(\mathbf{i_1}) \setminus \mathcal{K}_{c_1 - c_2\mathbf{i_1}}) \times_e (\mathcal{K}_{c_1 + c_2\mathbf{i_1}})$,

$$d(w_0, \mathcal{K}_{2,c}) > \frac{|z_{1,n} - z_{2,n}\mathbf{i_1}| \ln |z_{1,n} - z_{2,n}\mathbf{i_1}|}{2\sqrt{2}|z_{1,n} - z_{2,n}\mathbf{i_1}|^{\frac{1}{2^n}} |(z_{1,n} - z_{2,n}\mathbf{i_1})'|}$$

$$d(w_0, \mathcal{K}_{2,c}) < \frac{\sqrt{2}|z_{1,n} - z_{2,n}\mathbf{i_1}|}{|(z_{1,n} - z_{2,n}\mathbf{i_1})_n'|} \ln |z_{1,n} - z_{2,n}\mathbf{i_1}|$$

where $z_{1,n} - z_{2,n}\mathbf{i_1} := P_c^{\circ n}(z_1 - z_2\mathbf{i_1})$
and $(z_{1,n} - z_{2,n}\mathbf{i_1})' := \frac{d}{dz}[P_c^{\circ n}(z)]|_{z=z_1 - z_2\mathbf{i_1}}$.

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Approximated Distance Formulas

(**3**) If $w_0 \in (\mathcal{K}_{c_1-c_2\mathbf{i_1}}) \times_e (\mathbb{C}(\mathbf{i_1}) \setminus \mathcal{K}_{c_1+c_2\mathbf{i_1}})$.

-Similar to (**2**)-

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Ray-Tracing

Now we use the lower bound distance estimation formula D_l in conjunction with ray-tracing to produce images of bicomplex fractals.

Let $\vec{v}$ be an unitary vector in $\mathbb{R}^4$ and μ a point in $\mathbb{T}\backslash K_{2,c}$. Now define

$$\{Z_{\mu,\vec{v},n}\} := \begin{cases} Z_{\mu,\vec{v},0} = \mu \\ Z_{\mu,\vec{v},n} = Z_{\mu,\vec{v},n-1} + D_l(Z_{\mu,\vec{v},n-1})\vec{v} \end{cases}$$

By definition, no point in $K_{2,c}$ can be a member of such sequence.

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Ray-Tracing

If we set the projection eye to μ and use $\vec{v}$ as the orientation of our ray, then

$$\lim_{n\to\infty} Z_{\mu,\vec{v},n}$$

is our ray-tracing algorithm.

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Ray-Tracing

Two things may happen, we may miss the fractal or we may converge i.e.

$$\sum_{n=0}^{\infty} D_l(Z_{\mu,\vec{v},n}) = \infty$$

or

$$\sum_{n=0}^{\infty} D_l(Z_{\mu,\vec{v},n}) < \infty \implies \lim_{n\to\infty} D_l(Z_{\mu,\vec{v},n}) = 0.$$

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Exploration

The images of the fractals will be drawn on a screen, noted S that is defined by four coplanar points in space. These points are our screen corner. We divide S into pixel according to the resolution desired for our image. The position of the eye μ, will be function of the position and size of S. When we move S, μ will follow. We compute the first image of the object and while tracing the fractal, we keep stored the distance of the object.

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Exploration

To zoom into the region of interest, few steps are necessary. First we must recenter the selected region by rotating from μ, then resize the screen to the region of interest. Next, using a fraction of distance to the object, we move S forward. A typical implementation could use the ratio screen size region of interest of the fractal distance. As we get closer to the fractal, we should lower the ϵ value to keep a good level of fractal details.

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More details(images, movies and softwares) available on

$\boxed{\textbf{3dfractals.com}}$

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