Characterization of the Multicomplex Mandelbrot Set

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3D Mandelbrot Sets

- Quaternionic Mandelbrot set: Norton, 1982
- Bicomplex Mandelbrot set (Tetrabrot): Rochon, 2000
- Spherical coordinates (Mandelbulb, Power 8): White & Nylander, 2009

(a) 3D Mandelbrot sets

Bicomplex Numbers

Definition 1 ($\mathbb{M}(2)$ or $\mathbb{BC}$-space)

Let $z_1 = x_1 + x_2\mathbf{i_1}$, $z_2 = x_3 + x_4\mathbf{i_1}$ be two complex numbers $\mathbb{M}(1) \simeq \mathbb{C}$ with $\mathbf{i_1}^2 = -1$. A **bicomplex number** ζ is defined as:

$$\zeta = z_1 + z_2\mathbf{i_2}$$

where $\mathbf{i_2}^2 = -1$.

Various representations:

- In terms of four real numbers: $\zeta = x_1 + x_2\mathbf{i_1} + x_3\mathbf{i_2} + x_4\mathbf{j_1}$
- In terms of two idempotent elements:

$$\zeta = (z_1 - z_2\mathbf{i_1})\gamma_1 + (z_1 + z_2\mathbf{i_1})\overline{\gamma}_1$$

where $\gamma_1 = \frac{1+\mathbf{j_1}}{2}$ and $\overline{\gamma}_1 = \frac{1-\mathbf{j_1}}{2}$.

Operations on Bicomplex Numbers

Let $\zeta_1 = z_1 + z_2\mathbf{i_2}$ and $\zeta_2 = z_3 + z_4\mathbf{i_2}$.

1) Equality: $\zeta_1 = \zeta_2 \iff z_1 = z_3$ and $z_2 = z_4$
2) Addition: $\zeta_1 + \zeta_2 := (z_1 + z_3) + (z_2 + z_4)\mathbf{i_2}$
3) Multiplication: $\zeta_1 \cdot \zeta_2 := (z_1 z_3 - z_2 z_4) + (z_2 z_3 + z_1 z_4)\mathbf{i_2}$
4) Euclidean Norm: $|\zeta_1| = \sqrt{|z_1|^2 + |z_2|^2} = \sqrt{\sum_{i=1}^{4} x_i^2}$

Remark:

- $(\mathbb{M}(2), +, \cdot)$ forms a commutative ring with unity and zero divisors.
- $(\mathbb{M}(2), +, \cdot, |\cdot|)$ forms a **Banach space**.

Tricomplex Numbers

Definition 2 ($\mathbb{M}(3)$ or $\mathbb{TC}$-space)

Let $\zeta_1 = z_1 + z_2\mathbf{i_2}$, $\zeta_2 = z_3 + z_4\mathbf{i_2}$ be two bicomplex numbers. A **tricomplex number** η is defined as:

$$\eta = \zeta_1 + \zeta_2\mathbf{i_3}$$

where $\mathbf{i_3}^2 = -1$.

Various representations:

- In terms of four complex numbers: $\eta = z_1 + z_2\mathbf{i_2} + z_3\mathbf{i_3} + z_4\mathbf{j_3}$
- In terms of eight real numbers:

$$\eta = x_1 + x_2\mathbf{i_1} + x_3\mathbf{i_2} + x_4\mathbf{i_3} + x_5\mathbf{i_4} + x_6\mathbf{j_1} + x_7\mathbf{j_2} + x_8\mathbf{j_3}$$

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Tricomplex Numbers

Various representations (continuing):

- In terms of two idempotent elements:

$$\eta = (\zeta_1 - \zeta_2 \mathbf{i_2})\gamma_3 + (\zeta_1 + \zeta_2 \mathbf{i_2})\overline{\gamma}_3$$

  where $\zeta_1, \zeta_2 \in \mathbb{M}(2)$, $\gamma_3 = \frac{1+\mathbf{j_3}}{2}$ and $\overline{\gamma}_3 = \frac{1-\mathbf{j_3}}{2}$.

- In terms of four idempotent elements:

$$\eta = \eta_{\gamma_1\gamma_3} \cdot \gamma_1\gamma_3 + \eta_{\gamma_1\overline{\gamma}_3} \cdot \gamma_1\overline{\gamma}_3 + \eta_{\overline{\gamma}_1\gamma_3} \cdot \overline{\gamma}_1\gamma_3 + \eta_{\overline{\gamma}_1\overline{\gamma}_3} \cdot \overline{\gamma}_1\overline{\gamma}_3$$

  where $\eta_{\gamma_1\gamma_3}, \eta_{\gamma_1\overline{\gamma}_3}, \eta_{\overline{\gamma}_1\gamma_3}, \eta_{\overline{\gamma}_1\overline{\gamma}_3} \in \mathbb{M}(1) \simeq \mathbb{C}$ are defined as the **projections** in the plane.

Subsets of $\mathbb{M}(3)$

Definition 3

Let $\mathbf{i_k} \in \{\mathbf{i_1}, \mathbf{i_2}, \mathbf{i_3}, \mathbf{i_4}\}$ and $\mathbf{j_k} \in \{\mathbf{j_1}, \mathbf{j_2}, \mathbf{j_3}\}$, where $\mathbf{i_k^2} = -\mathbf{1}$ and $\mathbf{j_k^2} = \mathbf{1}$. We define

$$\mathbb{C}(\mathbf{i_k}) := \{x_0 + x_1\mathbf{i_k} \ : \ x_0, x_1 \in \mathbb{R}\}$$

and

$$\mathbb{D}(\mathbf{j_k}) := \{x_0 + x_1\mathbf{j_k} \ : \ x_0, x_1 \in \mathbb{R}\}.$$

- $\mathbb{C}(\mathbf{i_k})$ is a subset of $\mathbb{M}(3)$ for $k \in \{1, 2, 3, 4\}$. They are all isomorphic to $\mathbb{C}$. Notice that $\mathbb{C}(\mathbf{i_1}) = \mathbb{M}(1)$.
- $\mathbb{D}(\mathbf{j_k})$ is a subset of $\mathbb{M}(3)$ and is isomorphic to the set of **hyperbolic numbers** $\mathbb{D}$ for $k \in \{1, 2, 3\}$.

Subsets of $\mathbb{M}(3)$ (continuing)

Definition 4

Let $\mathbf{i_k}, \mathbf{i_l}, \mathbf{i_m} \in \{1, \mathbf{i_1}, \mathbf{i_2}, \mathbf{i_3}, \mathbf{i_4}, \mathbf{j_1}, \mathbf{j_2}, \mathbf{j_3}\}$ with $\mathbf{i_k} \neq \mathbf{i_l}$, $\mathbf{i_k} \neq \mathbf{i_m}$ and $\mathbf{i_l} \neq \mathbf{i_m}$. The third subset is

$$\mathbb{T}(\mathbf{i_m}, \mathbf{i_k}, \mathbf{i_l}) := \{x_1\mathbf{i_m} + x_2\mathbf{i_k} + x_3\mathbf{i_l} \ : \ x_1, x_2, x_3 \in \mathbb{R}\}.$$

- $\mathbb{T}(\mathbf{i_m}, \mathbf{i_k}, \mathbf{i_l}) = \text{span}_{\mathbb{R}}\{\mathbf{i_m}, \mathbf{i_k}, \mathbf{i_l}\}$.
- This sub-vector space of $\mathbb{M}(3)$ is used to make 3D slices in the tricomplex Mandelbrot set.

Multicomplex Numbers

More generally, the **multicomplex numbers** of order n (also called n-complex numbers) are obtained by using the previous duplication process recursively. They were first described by the Italian mathematician Corrado Segre in 1892. Indeed, for any integer $n \geq 1$, the set of multicomplex numbers of order n is defined as

$$\mathbb{M}(n) := \{\eta_1 + \eta_2 \mathbf{i_n} : \eta_1, \eta_2 \in \mathbb{M}(n-1)\}$$

with $\mathbf{i_n}^2 = -1$ and $\mathbb{M}(0) := \mathbb{R}$. Moreover, multicomplex addition and multiplication are defined similarly to the analogous complex operations, meaning that

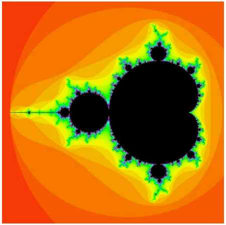
$$(\eta_1 + \eta_2 \mathbf{i_n}) + (\zeta_1 + \zeta_2 \mathbf{i_n}) = (\eta_1 + \zeta_1) + (\eta_2 + \zeta_2)\mathbf{i_n};$$
$$(\eta_1 + \eta_2 \mathbf{i_n})(\zeta_1 + \zeta_2 \mathbf{i_n}) = (\eta_1\zeta_1 - \eta_2\zeta_2) + (\eta_1\zeta_2 + \eta_2\zeta_1)\mathbf{i_n}.$$

The Mandelbrot Set

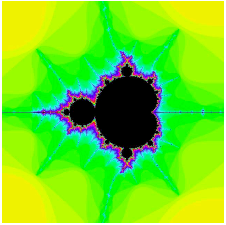
Definition 5

Let $Q_c(z) = z^2 + c$ a quadratic complex polynomial. The so-called Mandelbrot set is defined as follows:

$$\mathcal{M}^2 = \{c \in \mathbb{C} : \{Q_c^m(0)\}_{m=1}^{\infty} \text{ is bounded } \}.$$



(a) $\mathcal{M}^2$: Mandelbrot set



(b) $\mathcal{M}^2$: Zoom in

Tricomplex Mandelbrot Set: The Metatronbrot

Definition 6

Let $Q_c(\eta) = \eta^2 + c$ where $\eta, c \in \mathbb{M}(3)$. The **tricomplex Mandelbrot set** (also called *Metatronbrot*) is define as the set

$$\mathcal{M}_3^2 := \{c \in \mathbb{M}(3) \ : \ \{Q_c^m(0)\}_{m=1}^{\infty} \text{ is bounded } \}.$$

Theorem 7

A tricomplex number c is in $\mathcal{M}_3^2$ if and only if $|Q_c^m(0)| \leq 2$ for all natural number $m \geq 1$.

NOTE: The name **Metatronbrot** refers to the so-called **Metatron's cube** of the *Flower of Life*.

Principal 3D slices of $\mathcal{M}_3^2$

To visualize the 8D tricomplex Mandelbrot set, we have to define a **principal 3D slice** of $\mathcal{M}_3^2$.

$$\mathcal{T}^2 := \mathcal{T}^2(\mathbf{i_m}, \mathbf{i_k}, \mathbf{i_l}) = \{c \in \mathbb{T}(\mathbf{i_m}, \mathbf{i_k}, \mathbf{i_l}) \ : \ \{Q_c^m(0)\}_{m=1}^{\infty} \text{ is bounded } \}.$$

- There are 56 possible principal 3D slices.
- $\mathcal{T}^2(\mathbf{i_m}, \mathbf{i_k}, \mathbf{i_l}) = \mathcal{M}_3^2 \cap \mathbb{T}(\mathbf{i_m}, \mathbf{i_k}, \mathbf{i_l})$.
- The concept of **idempotent 3D slices** can be also defined using the idempotent basis.

Equivalence between principal 3D slices of $\mathcal{M}_3^2$

Definition 8

Let $\mathcal{T}_1^2(\mathbf{i_m}, \mathbf{i_k}, \mathbf{i_l})$ and $\mathcal{T}_2^2(\mathbf{i_n}, \mathbf{i_q}, \mathbf{i_s})$ be two principal 3D slices of the tricomplex Mandelbrot set $\mathcal{M}_3^2$. Then, $\mathcal{T}_1^2 \sim \mathcal{T}_2^2$ if we have a bijective linear mapping $\varphi : M_1 \to M_2$ such that $\varphi(\mathbb{T}_1(\mathbf{i_m}, \mathbf{i_k}, \mathbf{i_l})) = \mathbb{T}_2(\mathbf{i_n}, \mathbf{i_q}, \mathbf{i_s})$ and, for all $c \in \mathbb{T}_1(\mathbf{i_m}, \mathbf{i_k}, \mathbf{i_l})$

$$(\varphi \circ Q_c \circ \varphi^{-1})(\eta) = Q_{\varphi(c)}(\eta) \ \forall \eta \in M_2,$$

where M_i is the smallest sub-vector space of $\mathbb{M}(3)$ containing all iterates of $Q_{c_i}^m(0)$ with $c_i \in \mathbb{T}_i$ for $i = 1, 2$. In that case, we say that $\mathcal{T}_1^2$ and $\mathcal{T}_2^2$ have the **same dynamics**.

Principal Slices of $\mathcal{M}_3^2$

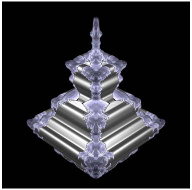
The number of principal 3D slices of the Metatronbrot $\mathcal{M}_3^2$ can be reduced to **eight slices**.

Theorem 9

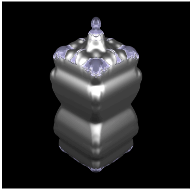
There are eight ***principal 3D slices*** *of the tricomplex Mandelbrot set* $\mathcal{M}_3^2$*:*

- $\mathcal{T}^2(1, \mathbf{i_1}, \mathbf{i_2})$ *called Tetrabrot;*
- $\mathcal{T}^2(\mathbf{i_1}, \mathbf{j_1}, \mathbf{j_2})$ *called Hourglassbrot;*
- $\mathcal{T}^2(1, \mathbf{j_1}, \mathbf{j_2})$ *called Airbrot;*
- $\mathcal{T}^2(\mathbf{i_1}, \mathbf{i_2}, \mathbf{i_3})$ *called Metabrot;*
- $\mathcal{T}^2(\mathbf{j_1}, \mathbf{j_2}, \mathbf{j_3})$ *called Firebrot;*
- $\mathcal{T}^2(\mathbf{i_1}, \mathbf{i_2}, \mathbf{j_1})$ *called Mousebrot;*
- $\mathcal{T}^2(\mathbf{i_1}, \mathbf{i_2}, \mathbf{j_2})$ *called Turtlebrot;*
- $\mathcal{T}^2(1, \mathbf{i_1}, \mathbf{j_1})$ *called Arrowheadbrot.*

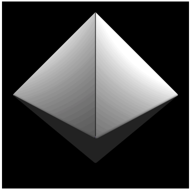
Family Shooting of the Metatronbrot: $\eta^2 + c$



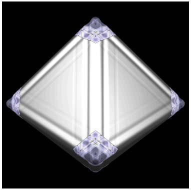
(a) Tetrabrot



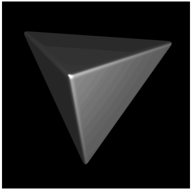
(b) Hourglassbrot



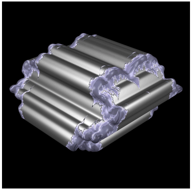
(c) Airbrot



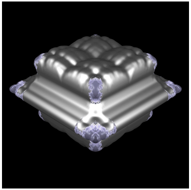
(d) Metabrot



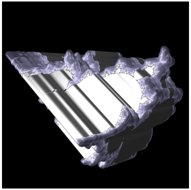
(e) Firebrot



(f) Mousebrot



(g) Turtlebrot



(h) Arrowheadbrot

Multicomplex Mandelbrot Set

We now consider the multicomplex case. Let $Q_{p,c}(\eta) = \eta^p + c$ and denote

$$Q^m_{p,c}(\eta) = \underbrace{(Q_{p,c} \circ Q_{p,c} \circ \cdots \circ Q_{p,c})}_{m \text{ times}}(\eta).$$

Using the function $Q_{p,c}$, we already defined the standard Mandelbrot set as

$$\mathcal{M}^2 = \left\{ c \in \mathbb{M}(1) \ : \ \{Q^m_{2,c}(0)\}^{\infty}_{m=1} \text{ is bounded } \right\}.$$

We can easily modify this last definition to obtain the following more general one.

Definition 10

Let $n, p \in \mathbb{N}$ such that $p \geq 2$. The n-complex **Multibrot** set of order p is defined as

$$\mathcal{M}^p_n = \left\{ c \in \mathbb{M}(n) \ : \ \{Q^m_{p,c}(0)\}^{\infty}_{m=1} \text{ is bounded } \right\}.$$

Characterization the 3D Slices

In the same way, we can generalized the previous concept of principal 3D slices into the multicomplexe spaces. In that case we obtain $\binom{2^n}{3}$ possible 3D slices. The next result gives a characterization of those slices.

Theorem 11

Let $\mathcal{T}_1^p$ be a principal 3D slice of $\mathcal{M}_n^p$. There always exists a tricomplex principal 3D slice $\mathcal{T}_2^p$ such that $\mathcal{T}_1^p \sim \mathcal{T}_2^p$ up to an affine transformation.

In other words, in that context, it is not necessary to explore principal 3D slices beyond the tricomplex space. Hence, the tricomplex space is, in a way, optimal.

Conclusion

In future works, it will therefore be possible to look into Multibrot principal slices specifically in the tricomplex case. In the specific case of the Metatronbrot $\mathcal{M}_3^2$, we know already that there are only eight principal 3D slices: the Tetrabrot, the Arrowheadbrot, the Hourglassbrot, the Airbrot, the Firebrot, the Mousebrot, the Metabrot and the Turtlebrot. Hence, for $p = 2$, these are the only principal 3D slices of the Mandelbrot set generalized to the multicomplex spaces.

Thanks for your attention!

(a) Hybridization between the Tetrabrot and a 3D kleinian IFS

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Table of imaginary units

$\cdot$	1	$\mathbf{i_1}$	$\mathbf{i_2}$	$\mathbf{i_3}$	$\mathbf{i_4}$	$\mathbf{j_1}$	$\mathbf{j_2}$	$\mathbf{j_3}$
1	1	$\mathbf{i_1}$	$\mathbf{i_2}$	$\mathbf{i_3}$	$\mathbf{i_4}$	$\mathbf{j_1}$	$\mathbf{j_2}$	$\mathbf{j_3}$
$\mathbf{i_1}$	$\mathbf{i_1}$	$-\mathbf{1}$	$\mathbf{j_1}$	$\mathbf{j_2}$	$-\mathbf{j_3}$	$-\mathbf{i_2}$	$-\mathbf{i_3}$	$\mathbf{i_4}$
$\mathbf{i_2}$	$\mathbf{i_2}$	$\mathbf{j_1}$	$-\mathbf{1}$	$\mathbf{j_3}$	$-\mathbf{j_2}$	$-\mathbf{i_1}$	$\mathbf{i_4}$	$-\mathbf{i_3}$
$\mathbf{i_3}$	$\mathbf{i_3}$	$\mathbf{j_2}$	$\mathbf{j_3}$	$-\mathbf{1}$	$-\mathbf{j_1}$	$\mathbf{i_4}$	$-\mathbf{i_1}$	$-\mathbf{i_2}$
$\mathbf{i_4}$	$\mathbf{i_4}$	$-\mathbf{j_3}$	$-\mathbf{j_2}$	$-\mathbf{j_1}$	$-\mathbf{1}$	$\mathbf{i_3}$	$\mathbf{i_2}$	$\mathbf{i_1}$
$\mathbf{j_1}$	$\mathbf{j_1}$	$-\mathbf{i_2}$	$-\mathbf{i_1}$	$\mathbf{i_4}$	$\mathbf{i_3}$	$\mathbf{1}$	$-\mathbf{j_3}$	$-\mathbf{j_2}$
$\mathbf{j_2}$	$\mathbf{j_2}$	$-\mathbf{i_3}$	$\mathbf{i_4}$	$-\mathbf{i_1}$	$\mathbf{i_2}$	$-\mathbf{j_3}$	$\mathbf{1}$	$-\mathbf{j_1}$
$\mathbf{j_3}$	$\mathbf{j_3}$	$\mathbf{i_4}$	$-\mathbf{i_3}$	$-\mathbf{i_2}$	$\mathbf{i_1}$	$-\mathbf{j_2}$	$-\mathbf{j_1}$	$\mathbf{1}$

Table: Product of tricomplex imaginary units

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