On a Tricomplex Distance Estimation for Generalized Multibrot Sets

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**Definition 1 (M(2) or BC-space)**

Let $z_1 = x_1 + x_2 i_1$, $z_2 = x_3 + x_4 i_1$ be two complex numbers $\mathbb{M}(1) \cong \mathbb{C}$ with $i_1^2 = -1$. A **bicomplex number** $\zeta$ is defined as:

$$\zeta = z_1 + z_2 i_2$$

(1)

where $i_2^2 = -1$.

**Various representations:**

- **In terms of four real numbers:** $\zeta = x_1 + x_2 i_1 + x_3 i_2 + x_4 j_1$
- **In terms of two idempotent elements:**

$$\zeta = (z_1 - z_2 i_1) \gamma_1 + (z_1 + z_2 i_1) \overline{\gamma}_1$$

where $\gamma_1 = \frac{1+j_1}{2}$ and $\overline{\gamma}_1 = \frac{1-j_1}{2}$. 

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Operations on Bicomplex Numbers

Let \( \zeta_1 = z_1 + z_2 i_2 \) and \( \zeta_2 = z_3 + z_4 i_2 \).

1) Equality: \( \zeta_1 = \zeta_2 \iff z_1 = z_3 \) and \( z_2 = z_4 \).

2) Addition: \( \zeta_1 + \zeta_2 := (z_1 + z_3) + (z_2 + z_4)i_2 \).

3) Multiplication: \( \zeta_1 \cdot \zeta_2 := (z_1z_3 - z_2z_4) + (z_2z_3 + z_1z_4)i_2 \).

4) Euclidean Norm: \( |\zeta_1| = \sqrt{|z_1|^2 + |z_2|^2} = \sqrt{\sum_{i=1}^{4} x_i^2} \)

Remark:

- \((\mathbb{M}(2), +, \cdot)\) forms a commutative ring with unity and zero divisors.
- \((\mathbb{M}(2), +, \cdot, |\cdot|)\) forms a Banach space.
Tricomplex Numbers

Definition 2 ($\mathbb{M}(3)$ or $\mathbb{TC}$-space)

Let $\zeta_1 = z_1 + z_2i_2$, $\zeta_2 = z_3 + z_4i_2$ be two bicomplex numbers. A tricomplex number $\eta$ is defined as:

$$\eta = \zeta_1 + \zeta_2i_3$$

(2)

where $i_3^2 = -1$.

Various representations:

- In terms of four complex numbers: $\eta = z_1 + z_2i_2 + z_3i_3 + z_4j_3$
- In terms of eight real numbers:

$$\eta = x_1 + x_2i_1 + x_3i_2 + x_4i_3 + x_5i_4 + x_6j_1 + x_7j_2 + x_8j_3$$
Various representations (continuing):

- In terms of two idempotent elements:

\[ \eta = (\zeta_1 - \zeta_2 i_2) \gamma_3 + (\zeta_1 + \zeta_2 i_2) \overline{\gamma}_3 \]

where \( \zeta_1, \zeta_2 \in \mathbb{M}(2) \), \( \gamma_3 = \frac{1+i_3}{2} \) and \( \overline{\gamma}_3 = \frac{1-i_3}{2} \).

- In terms of four idempotent elements:

\[ \eta = \eta_{\gamma_1 \gamma_3} \cdot \gamma_1 \gamma_3 + \eta_{\gamma_1 \overline{\gamma}_3} \cdot \gamma_1 \overline{\gamma}_3 + \eta_{\overline{\gamma}_1 \gamma_3} \cdot \gamma_1 \gamma_3 + \eta_{\overline{\gamma}_1 \overline{\gamma}_3} \cdot \gamma_1 \overline{\gamma}_3 \]

where \( \eta_{\gamma_1 \gamma_3}, \eta_{\gamma_1 \overline{\gamma}_3}, \eta_{\overline{\gamma}_1 \gamma_3}, \eta_{\overline{\gamma}_1 \overline{\gamma}_3} \in \mathbb{M}(1) \cong \mathbb{C} \) are defined as the projections in the plane.
Subsets of $\mathbb{M}(3)$

**Definition 3**

Let $i_k \in \{i_1, i_2, i_3, i_4\}$ and $j_k \in \{j_1, j_2, j_3\}$, where $i_k^2 = -1$ and $j_k^2 = 1$. We define

$$C(i_k) := \{\eta = x_0 + x_1 i_k : x_0, x_1 \in \mathbb{R}\}$$

and

$$D(j_k) := \{x_0 + x_1 j_k : x_0, x_1 \in \mathbb{R}\}.$$  

- $C(i_k)$ is a subset of $\mathbb{M}(3)$ for $k \in \{1, 2, 3, 4\}$. They are all isomorphic to $\mathbb{C}$. Notice that $C(i_1) = \mathbb{M}(1)$.
- $D(j_k)$ is a subset of $\mathbb{M}(3)$ and is isomorphic to the set of hyperbolic numbers $\mathbb{D}$ for $k \in \{1, 2, 3\}$. 
Definition 4

Let $i_k, i_l, i_m \in \{1, i_1, i_2, i_3, i_4, j_1, j_2, j_3\}$ with $i_k \neq i_l, i_k \neq i_m$ and $i_l \neq i_m$. The third subset is

$$T(i_m, i_k, i_l) := \{x_1i_m + x_2i_k + x_3i_l : x_1, x_2, x_3 \in \mathbb{R}\}.$$  \hspace{1cm} (3)

- $T(i_m, i_k, i_l) = \text{span}_\mathbb{R}\{i_m, i_k, i_l\}$.
- This sub-vector space of $\mathbb{M}(3)$ is used to make 3D slices in the tricomplex multibrot sets.
Definition of Multibrots in the complex plane

Definition 5

Let $Q_{p,c}(z) = z^p + c$ a polynomial of degree $p \in \mathbb{N} \setminus \{0, 1\}$. A Multibrot set is the set of complex numbers $c$ for which the sequence $\{Q_{p,c}^m(0)\}_{m=1}^{\infty}$ is bounded, i.e.

$$\mathcal{M}^p = \left\{ c \in \mathbb{C} : \{Q_{p,c}^m(0)\}_{m=1}^{\infty} \text{ is bounded} \right\}. \quad (4)$$

- If we set $p = 2$, we find the well-known Mandelbrot set.
Properties of Multibrot sets

**Theorem 6**

*For all complex number* $c$ *in* $\mathcal{M}^p$, *we have* $|c| \leq 2^{1/(p-1)}$.

**Theorem 7**

*A complex number* $c$ *is in* $\mathcal{M}^p$ *if and only if* $|Q_{p,c}^m(0)| \leq 2^{1/(p-1)}$ *for all natural number* $m \geq 1$.

- For an integer $p \geq 2$, the set $\mathcal{M}^p$ is contained a closed discus in $\mathbb{C}$.
- Theorem 7 gives a method to visualize the Multibrot sets.
Multibrot sets Pictured

(a) $\mathcal{M}^2$: Mandelbrot set

(b) $\mathcal{M}^2$: Zoom in
Multibrot sets Pictured

(a) $\mathcal{M}^3$: Mandelbric set

(b) $\mathcal{M}^3$: Zoom in
Multibrot sets Pictured

(a) $M^6$

(b) $M^6$: Zoom in

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Tricomplex Multibrot Sets

Definition 8

Let $Q_{p,c}(\eta) = \eta^p + c$ where $\eta, c \in \mathbb{M}(3)$ and $p \geq 2$ an integer. The tricomplex Multibrot of order $p$ is defined as the set

$$\mathcal{M}_3^p := \left\{ c \in \mathbb{M}(3) : \{ Q_{p,c}^m(0) \}_{m=1}^{\infty} \text{ is bounded} \right\}. \quad (5)$$

Theorem 9

A tricomplex number $c$ is in $\mathcal{M}_3^p$ if and only if $|Q_{p,c}^m(0)| \leq 2^{1/(p-1)}$ for all natural number $m \geq 1$. 

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Tricomplex Multibrot Sets

From the idempotent representations, we can define the $\mathbb{T}\mathbb{C}$-Cartesian product as

$$X_1 \times \gamma_3 X_2 := \{x_1 \gamma_3 + x_2 \overline{\gamma}_3 : x_1 \in X_1, x_2 \in X_2\}$$

where $X_1, X_2 \subset \mathbb{B}\mathbb{C}$. Moreover, we have the following $\mathbb{B}\mathbb{C}$-Cartesian product

$$X_1 \times \gamma_1 X_2 := \{x_1 \gamma_1 + x_2 \overline{\gamma}_1 : x_1 \in X_1, x_2 \in X_2\}$$

where $X_1, X_2 \subset \mathbb{C}(i_1)$.

**Theorem 10**

$$\mathcal{M}_3^p = (\mathcal{M}^p \times \gamma_1 \mathcal{M}^p) \times \gamma_3 (\mathcal{M}^p \times \gamma_1 \mathcal{M}^p).$$
To visualize the tricomplex multibrot sets, we have to define a principal 3D slice of $\mathcal{M}_3$.

$$\mathcal{T}^p := \mathcal{T}^p(i_m, i_k, i_l) = \left\{ c \in \mathbb{T}(i_m, i_k, i_l) : \{ Q_{p,c}^m(0) \}_{m=1}^\infty \text{ is bounded} \right\}.$$

There are 56 possible 3D principal slices.
Definition 11

Let $T_1^p(i_m, i_k, i_l)$ and $T_2^p(i_n, i_q, i_s)$ be two principal 3D slices of a tricomplex Multibrot set $M_3^P$. Then, $T_1^p \sim T_2^p$ if we have a bijective linear mapping $\varphi : M(3) \to M(3)$ such that $\forall c_2 \in T(i_n, i_q, i_s)$ there exists a $c_1 \in T(i_m, i_k, i_l)$ with $\varphi(c_1) = c_2$ and

$$(\varphi \circ Q_{p,c_1} \circ \varphi^{-1})(\eta) = Q_{p,c_2}(\eta) \ \forall \eta \in M(3).$$

In that case, we say that $T_1^p$ and $T_2^p$ have the same dynamics.
The number of principal 3D slices of the set $M^2_3$ can be reduced to 8 slices.

**Theorem 12**

There are eight principal 3D slices of the tricomplex multibrot set $M^2_3$:

- $T^2(1, i_1, i_2)$ called Tetrabrot;
- $T^2(i_1, j_1, j_2)$ called Hourglassbrot;
- $T^2(1, j_1, j_2)$ called Perplexbrot;
- $T^2(i_1, i_2, i_3)$ called Metabrot.
- $T^2(j_1, j_2, j_3)$ called Firebrot;
- $T^2(i_1, i_2, j_1)$ called Mousebrot;
- $T^2(i_1, i_2, j_2)$ called Turtlebrot;
- $T^2(1, i_1, j_1)$ called Arrow-Pitbrot.
Family Shooting: square $\eta^2 + c$

(a) Tetrabrot  (b) Hourglassbrot  (c) Perplexbrot  (d) Metabrot

(e) Firebrot  (f) Mousebrot  (g) Turtlebrot  (h) Arrow-Pitbrot
The number of principal 3D slices of the set $\mathcal{M}_3^3$ can be reduced to only 4 slices!

**Theorem 13**

*There are four principal 3D slices of the tricomplex multibrot set $\mathcal{M}_3^3$:*

- $\mathcal{T}^3(1, i_1, i_2)$ called Tetrabric;
- $\mathcal{T}^3(1, j_1, j_2)$ called Perplexbric;
- $\mathcal{T}^3(1, i_1, j_1)$ called Hourglassbric;
- $\mathcal{T}^3(i_1, i_2, i_3)$ called Metabric.*
Family Shooting: cubic $\eta^3 + c$

- (a) Tetrabric
- (b) Perplexbric
- (c) Hourglassbric
- (d) Metabric
To explore these sets deeply, we need to use the ray tracing technique. We have the following theorem for the bounds of the distance from a point $c \in \mathbb{C}(i_1) \setminus \mathcal{M}^p$ to the set $\mathcal{M}^p$.

**Theorem 14**

Let $c \in \mathbb{C}(i_1) \setminus \mathcal{M}^p$ and define $d(c, \mathcal{M}^p) := \inf \{|z - c| : z \in \mathcal{M}^p\}$. Then,

$$\frac{\sinh(G(c))}{2e^{G(c)}|G'(c)|} < d(c, \mathcal{M}^p) < \frac{2\sinh(G(c))}{|G'(c)|}$$

where $G$ is the Green’s function of the set $\mathcal{M}^p$. 

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Main Result

We proposed to use this approximation formula, for $p \geq 2$, in the plane.

Conjecture 1

The distance $d(c, \mathcal{M}^p)$ from a point $c \notin \mathcal{M}^p$ to the set $\mathcal{M}^p$ can be approximated in the following way:

$$\frac{p|c_m| \ln |c_m|}{2|c_m|^{1/p^m}|c'_m|} < d(c, \mathcal{M}^p)$$

where $c_m := Q^m_{p,c}(0)$, and $c'_m := \frac{d}{dc} (Q^m_{p,c}(0))\big|_{c=c_0}$. 

Main Result

In the tricomplex space, the key result is the following. Define, for \( \eta_0 \not\in X \),

\[
d(\eta', X) := \inf \{ |\eta - \eta'| : \eta \in X \}
\]

which give the distance between the point \( \eta' \) and the set \( X \subset \mathbb{T}C \).

**Theorem 15**

If \( X \subset \mathbb{T}C \) is a compact set, and

\[
X = (X_{\gamma_1 \gamma_3} \times \gamma_1 X_{\bar{\gamma_1} \gamma_3}) \times \gamma_3 (X_{\gamma_1 \gamma_3} \times \gamma_1 X_{\bar{\gamma_1} \gamma_3}),
\]

then

\[
d(\eta', X) = \sqrt{\frac{d(\eta'_{\gamma_1 \gamma_3}, X_{\gamma_1 \gamma_3})^2 + d(\eta'_{\bar{\gamma_1} \gamma_3}, X_{\bar{\gamma_1} \gamma_3})^2 + d(\eta'_{\gamma_1 \gamma_3}, X_{\bar{\gamma_1} \gamma_3})^2 + d(\eta'_{\bar{\gamma_1} \gamma_3}, X_{\gamma_1 \gamma_3})^2}{4}}.
\]
Houglassbrot Exploration

We are now able to use the approximation formula of the complex plane in each idempotent components of a tricomplex multibrot set.

\[ T^2(i_1, j_1, j_2) \] [YouTube Hyperlink]
References


### Table of imaginary units

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**Table:** Product of tricomplex imaginary units