

Infinite-Dimensional Bicomplex Hilbert Spaces

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- Bicomplex numbers, just like quaternions, are a generalization of complex numbers by means of entities specified by four real numbers. These two number systems, however, are different in two important ways: quaternions, which form a division algebra, are noncommutative, whereas bicomplex numbers are commutative but do not form a division algebra.
- Division algebras do not have zero divisors, that is, nonzero elements whose product is zero. Many believe that any attempt to generalize quantum mechanics to number systems other than complex numbers should retain the division algebra property. Indeed considerable work has been done over the years on quaternionic quantum mechanics.

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- In the past few years, however, it was pointed out that several features of quantum mechanics can be generalized to bicomplex numbers. A generalization of Schrödinger's equation for a particle in one dimension was proposed, and self-adjoint operators were defined on finite-dimensional bicomplex Hilbert spaces. Recently, eigenvalues and eigenfunctions of the bicomplex analogue of the quantum harmonic oscillator Hamiltonian were obtained in full generality.
- The perspective of generalizing quantum mechanics to bicomplex numbers motivates us in developing further mathematical tools related to infinite-dimensional bicomplex Hilbert spaces and operators acting on them.

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- The perspective of generalizing quantum mechanics to bicomplex numbers motivates us in developing further mathematical tools related to infinite-dimensional bicomplex Hilbert spaces and operators acting on them.

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- 4 The Bicomplex Quantum Mechanics
 - The Harmonic Oscillator

Definition

Bicomplex numbers are defined as

$$\mathbb{T} := \{z_1 + z_2 \mathbf{i}_2 \mid z_1, z_2 \in \mathbb{C}(\mathbf{i}_1)\}$$

where the imaginary units $\mathbf{i}_1, \mathbf{i}_2$ and \mathbf{j} are governed by the rules:
 $\mathbf{i}_1^2 = \mathbf{i}_2^2 = -1$, $\mathbf{j}^2 = 1$ and

$$\begin{aligned} \mathbf{i}_1 \mathbf{i}_2 &= \mathbf{i}_2 \mathbf{i}_1 = \mathbf{j}, \\ \mathbf{i}_1 \mathbf{j} &= \mathbf{j} \mathbf{i}_1 = -\mathbf{i}_2, \\ \mathbf{i}_2 \mathbf{j} &= \mathbf{j} \mathbf{i}_2 = -\mathbf{i}_1. \end{aligned}$$

- Note that we define $\mathbb{C}(\mathbf{i}_k) := \{x + y \mathbf{i}_k \mid \mathbf{i}_k^2 = -1 \text{ and } x, y \in \mathbb{R}\}$ for $k = 1, 2$.

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In fact, the bicomplex numbers

$$\mathbb{T} \cong \text{Cl}_{\mathbb{C}}(1, 0) \cong \text{Cl}_{\mathbb{C}}(0, 1)$$

are *unique* among the complex Clifford algebras in that they are commutative but not division algebra. It is also convenient to write the set of bicomplex numbers as

$$\mathbb{T} := \{w_0 + w_1 \mathbf{i}_1 + w_2 \mathbf{i}_2 + w_3 \mathbf{j} \mid w_0, w_1, w_2, w_3 \in \mathbb{R}\}.$$

- In particular, if we put $z_1 = x$ and $z_2 = y \mathbf{i}_1$ with $x, y \in \mathbb{R}$ in $z_1 + z_2 \mathbf{i}_2$, then we obtain the following subalgebra of hyperbolic numbers, also called duplex numbers:

$$\mathbb{D} := \{x + y \mathbf{j} \mid \mathbf{j}^2 = 1, x, y \in \mathbb{R}\} \cong \text{Cl}_{\mathbb{R}}(0, 1).$$

- Zero divisors make up the so-called null cone \mathcal{NC} . That terminology comes from the fact that when w is written as $z_1 + z_2 \mathbf{i}_2$, zero divisors are such that $z_1^2 + z_2^2 = 0$.

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- Complex conjugation plays an important role both for algebraic and geometric properties of \mathbb{C} . For bicomplex numbers, there are three possible conjugations. Let $w \in \mathbb{T}$ and $z_1, z_2 \in \mathbb{C}(\mathbf{i}_1)$ such that $w = z_1 + z_2\mathbf{i}_2$. Then we define the three conjugations as:

$$w^{\dagger_1} = (z_1 + z_2\mathbf{i}_2)^{\dagger_1} := \bar{z}_1 + \bar{z}_2\mathbf{i}_2,$$

$$w^{\dagger_2} = (z_1 + z_2\mathbf{i}_2)^{\dagger_2} := z_1 - z_2\mathbf{i}_2,$$

$$w^{\dagger_3} = (z_1 + z_2\mathbf{i}_2)^{\dagger_3} := \bar{z}_1 - \bar{z}_2\mathbf{i}_2,$$

where \bar{z}_k is the standard complex conjugate of complex numbers $z_k \in \mathbb{C}(\mathbf{i}_1)$.

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We know that the product of a standard complex number with its conjugate gives the square of the Euclidean metric in \mathbb{R}^2 . The analogs of this, for bicomplex numbers, are the following. Let $z_1, z_2 \in \mathbb{C}(\mathbf{i}_1)$ and $w = z_1 + z_2 \mathbf{i}_2 \in \mathbb{T}$, then we have that:

$$|w|_{\mathbf{i}_1}^2 := w \cdot w^{\dagger_2} \in \mathbb{C}(\mathbf{i}_1),$$

$$|w|_{\mathbf{i}_2}^2 := w \cdot w^{\dagger_1} \in \mathbb{C}(\mathbf{i}_2),$$

$$|w|_{\mathbf{j}}^2 := w \cdot w^{\dagger_3} \in \mathbb{D}.$$

In this talk we will often use the Euclidean \mathbb{R}^4 norm defined as

$$|w| := \sqrt{|z_1|^2 + |z_2|^2} = \sqrt{\operatorname{Re}(|w|_{\mathbf{j}}^2)}.$$

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- It is also important to know that every bicomplex number $w = z_1 + z_2\mathbf{i}_2$ has the following unique idempotent representation:

$$z_1 + z_2\mathbf{i}_2 = (z_1 - z_2\mathbf{i}_1)\mathbf{e}_1 + (z_1 + z_2\mathbf{i}_1)\mathbf{e}_2.$$

where $\mathbf{e}_1 = \frac{1+\mathbf{j}}{2}$ and $\mathbf{e}_2 = \frac{1-\mathbf{j}}{2}$.

- From this, we can introduce two projection operators

$$P_1 : (z_1 + z_2\mathbf{i}_2) \in \mathbb{T} \mapsto (z_1 + z_2\mathbf{i}_2)_{\hat{1}} \in \mathbb{C}(\mathbf{i}_1),$$

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- The projection operators P_k have the interesting properties

$$[P_k]^2 = P_k, \quad P_1\mathbf{e}_1 + P_2\mathbf{e}_2 = \text{Id}, \quad k = 1, 2,$$

and for $s, t \in \mathbb{T}$, we have

$$P_k(s + t) = P_k(s) + P_k(t), \quad P_k(s \cdot t) = P_k(s) \cdot P_k(t), \quad k = 1, 2.$$

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- Bicomplex numbers make up a commutative ring. What vector spaces are to fields, modules are to rings. A module defined over the ring \mathbb{T} of bicomplex numbers will be called a \mathbb{T} -module.

Definition

Let M be a \mathbb{T} -module. For $k = 1, 2$, we define V_k as the set of all elements of the form $\mathbf{e}_k|\psi\rangle$, with $|\psi\rangle \in M$. Succinctly, $V_1 := \mathbf{e}_1 M$ and $V_2 := \mathbf{e}_2 M$.

- We have used Dirac's notation for elements of M which, following, we will call *kets*.
- For $k = 1, 2$, addition and multiplication by a $\mathbb{C}(\mathbf{i}_1)$ scalar are closed in V_k . Therefore, V_k is a vector space over $\mathbb{C}(\mathbf{i}_1)$. Any element $|\psi\rangle_k \in V_k$ satisfies $|\psi\rangle_k = \mathbf{e}_k|\psi\rangle_k$.

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Theorem

The \mathbb{T} -module M can be viewed as a vector space M' over $\mathbb{C}(\mathbf{i}_1)$, and $M' = V_1 \oplus V_2$.

- From a set-theoretical point of view, M and M' are identical. In this sense we can say, perhaps improperly, that the **module** M can be decomposed into the direct sum of two vector spaces over $\mathbb{C}(\mathbf{i}_1)$, i.e. $M = V_1 \oplus V_2$.
- Henceforth we will write $|\psi\rangle_{\mathbf{k}} = \mathbf{e}_{\mathbf{k}}|\psi\rangle$, keeping in mind that $\mathbf{e}_{\mathbf{k}}|\psi\rangle_{\mathbf{k}} = |\psi\rangle_{\mathbf{k}} \in V_{\mathbf{k}}$ for $\mathbf{k} = 1, 2$.

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- The norm of a vector is an important concept in vector space theory. We will now generalize it to \mathbb{T} -modules, making use of the association established in the last Theorem.

definition

Let M be a \mathbb{T} -module and let M' be the associated vector space. We say that $\|\cdot\| : M \rightarrow \mathbb{R}$ is a **\mathbb{T} -norm** on M if the following holds:

1. $\|\cdot\| : M' \rightarrow \mathbb{R}$ is a norm;
2. $\|w \cdot |\psi\rangle\| \leq \sqrt{2}|w| \cdot \||\psi\rangle\|$, $\forall w \in \mathbb{T}$, $\forall |\psi\rangle \in M$.

- A \mathbb{T} -module with a **\mathbb{T} -norm** is called a **normed \mathbb{T} -module**.

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Definition

Let M be a \mathbb{T} -module. Suppose that with each pair $|\psi\rangle$ and $|\phi\rangle$ in M , taken in this order, we associate a bicomplex number $(|\psi\rangle, |\phi\rangle)$. We say that the association defines a bicomplex scalar (or inner) product if it satisfies the following conditions:

1. $(|\psi\rangle, |\phi\rangle + |\chi\rangle) = (|\psi\rangle, |\phi\rangle) + (|\psi\rangle, |\chi\rangle), \forall |\psi\rangle, |\phi\rangle, |\chi\rangle \in M;$
2. $(|\psi\rangle, \alpha|\phi\rangle) = \alpha(|\psi\rangle, |\phi\rangle), \forall \alpha \in \mathbb{T}, \forall |\psi\rangle, |\phi\rangle \in M;$
3. $(|\psi\rangle, |\phi\rangle) = (|\phi\rangle, |\psi\rangle)^{\dagger_3}, \forall |\psi\rangle, |\phi\rangle \in M;$
4. $(|\psi\rangle, |\psi\rangle) = 0 \Leftrightarrow |\psi\rangle = 0, \forall |\psi\rangle \in M.$

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1. $(|\psi\rangle, |\phi\rangle + |\chi\rangle) = (|\psi\rangle, |\phi\rangle) + (|\psi\rangle, |\chi\rangle), \forall |\psi\rangle, |\phi\rangle, |\chi\rangle \in M;$
2. $(|\psi\rangle, \alpha|\phi\rangle) = \alpha(|\psi\rangle, |\phi\rangle), \forall \alpha \in \mathbb{T}, \forall |\psi\rangle, |\phi\rangle \in M;$
3. $(|\psi\rangle, |\phi\rangle) = (|\phi\rangle, |\psi\rangle)^{\dagger 3}, \forall |\psi\rangle, |\phi\rangle \in M;$
4. $(|\psi\rangle, |\psi\rangle) = 0 \Leftrightarrow |\psi\rangle = 0, \forall |\psi\rangle \in M.$

- Property 3 implies that $(|\psi\rangle, |\psi\rangle) \in \mathbb{D}$. Definition 3 is intended to be very general. In this paper we shall be more restrictive, by requiring the bicomplex scalar product (\cdot, \cdot) to be *hyperbolic positive*, that is,

$$(|\psi\rangle, |\psi\rangle) \in \mathbb{D}^+, \forall |\psi\rangle \in M.$$

- From the last definition it is easy to see that the following projection of a bicomplex scalar product:

$$(\cdot, \cdot)_{\widehat{k}} := P_k((\cdot, \cdot)) : M \times M \longrightarrow \mathbb{C}(\mathbf{i}_k)$$

is a **standard scalar product** on V_k , for $k = 1, 2$.

Theorem

Let $|\psi\rangle, |\phi\rangle \in M$, then

$$(|\psi\rangle, |\phi\rangle) = \mathbf{e}_1(|\psi\rangle_1, |\phi\rangle_1)_{\widehat{1}} + \mathbf{e}_2(|\psi\rangle_2, |\phi\rangle_2)_{\widehat{2}}.$$

Moreover, the bicomplex scalar product is **completely characterized** by the two standard scalar products $(\cdot, \cdot)_{\widehat{k}}$ on V_k .

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- In this section we define the notion of a bicomplex Hilbert space and prove the analog of the Riesz representation theorem.

Definition

Let M be a \mathbb{T} -module and let (\cdot, \cdot) be a bicomplex scalar product defined on M . The space $\{M, (\cdot, \cdot)\}$ is called a \mathbb{T} -inner product space, or bicomplex pre-Hilbert space. When no confusion arises, $\{M, (\cdot, \cdot)\}$ will simply be denoted by M .

- If V_1 and V_2 are complete, then $M' = V_1 \oplus V_2$ is a direct sum of two Hilbert spaces. It is easy to see that M' is also a Hilbert space, when the following natural scalar product is defined over the direct sum:

$$(|\psi\rangle_1 \oplus |\psi\rangle_2, |\phi\rangle_1 \oplus |\phi\rangle_2) = (|\psi\rangle_1, |\phi\rangle_1)_{\hat{1}} + (|\psi\rangle_2, |\phi\rangle_2)_{\hat{2}}.$$

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- From this scalar product, we can define a **norm** on the vector space M' :

$$\begin{aligned} |||\phi\rangle|| &:= \frac{1}{\sqrt{2}} \sqrt{(|\phi\rangle_1, |\phi\rangle_1)_{\hat{1}} + (|\phi\rangle_2, |\phi\rangle_2)_{\hat{2}}} \\ &= \frac{1}{\sqrt{2}} \sqrt{||\phi\rangle_1|_1|^2 + ||\phi\rangle_2|_2|^2}. \end{aligned} \quad (1)$$

Here we wrote

$$||\phi\rangle_{\mathbf{k}}|_{\mathbf{k}} = \sqrt{(|\phi\rangle_{\mathbf{k}}, |\phi\rangle_{\mathbf{k}})_{\hat{\mathbf{k}}}},$$

where $|\cdot|_{\mathbf{k}}$ is the natural scalar-product-induced norm on $V_{\mathbf{k}}$. The $1/\sqrt{2}$ factor in (1) is introduced so as to relate in a simple manner the norm with the bicomplex scalar product.

- Indeed we have

$$\| |\phi\rangle \| = \frac{1}{\sqrt{2}} \sqrt{(|\phi\rangle_1, |\phi\rangle_1)_{\hat{1}} + (|\phi\rangle_2, |\phi\rangle_2)_{\hat{2}}} = |\sqrt{(|\phi\rangle, |\phi\rangle)}|.$$

- It is easy to check that $\| \cdot \|$ is a \mathbb{T} -norm on M and that the \mathbb{T} -module M is **complete** with respect to the following metric on M :

$$d(|\phi\rangle, |\psi\rangle) = \| |\phi\rangle - |\psi\rangle \|.$$

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- Let us summarize what we found by means of a definition and a theorem.

Definition

A bicomplex Hilbert space is a \mathbb{T} -inner product space M which is complete with respect to the induced \mathbb{T} -norm (1).

Theorem

Let M be a bicomplex Hilbert space. Then $(V_k, (\cdot, \cdot)_{\widehat{k}})$ is a complex (in $\mathbb{C}(\mathbf{i}_1)$) Hilbert space for $k = 1, 2$.

- As a direct application of this result, we obtain the following **Bicomplex Riesz Representation Theorem**.

Theorem (Riesz)

Let $\{M, (\cdot, \cdot)\}$ be a bicomplex Hilbert space and let $f : M \rightarrow \mathbb{T}$ be a continuous linear functional on M . Then there is a unique $|\psi\rangle \in M$ such that $\forall |\phi\rangle \in M, f(|\phi\rangle) = (|\psi\rangle, |\phi\rangle)$.

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- We close this section by proving a general version of Schwarz's inequality in a bicomplex Hilbert space.

Theorem (Bicomplex Schwarz inequality)

Let $|\psi\rangle, |\phi\rangle \in M$. Then

$$|(|\psi\rangle, |\phi\rangle)| \leq \sqrt{2} \|\psi\| \|\phi\|.$$

- This result is a direct generalization of the bicomplex Schwarz inequality obtained for finite bicomplex Hilbert space and proved here for an arbitrary bicomplex Hilbert space M without any extra condition on the bicomplex scalar product.

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- In this section we investigate more specific \mathbb{T} -modules, namely those that have a countable basis.

Definition

Let M be a normed \mathbb{T} -module. We say that M has a **Schauder \mathbb{T} -basis** if there exists a countable set $\{|m_1\rangle \dots |m_l\rangle \dots\}$ of elements of M such that every element $|\psi\rangle \in M$ admits a unique decomposition as the sum of a convergent series $|\psi\rangle = \sum_{l=1}^{\infty} w_l |m_l\rangle$, $w_l \in \mathbb{T}$.

- If $\{|m_l\rangle\}$ is a Schauder \mathbb{T} -basis in M , it follows that $\sum_{l=1}^{\infty} w_l |m_l\rangle = 0$ if and only if $w_l = 0$, $\forall l \in \mathbb{N}$. Moreover, if a normed \mathbb{T} -module M has a **Schauder \mathbb{T} -basis**, then the vector space M' is automatically a normed vector space with the following classical Schauder basis:

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- The normed space M' with a Schauder basis is necessarily of infinite dimension since it contains subspaces of an arbitrary finite dimension.
- A normed \mathbb{T} -module with a Schauder \mathbb{T} -basis is called a **countable \mathbb{T} -module**. For the rest of this section we will only consider bicomplex Hilbert spaces constructed from countable \mathbb{T} -modules. We now show that in this context, it is always possible to construct an orthonormal Schauder \mathbb{T} -basis in M .

Theorem (Orthonormalization)

Let M be a bicomplex Hilbert space and let $\{|s_j\rangle\}$ be an arbitrary Schauder \mathbb{T} -basis of M . Then $\{|s_j\rangle\}$ can always be orthonormalized.

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Let $\{|\psi_n\rangle\}$ be an orthonormal sequence in the bicomplex Hilbert space M and let $\{\alpha_n\}$ be a sequence of bicomplex numbers. Then the series $\sum_{n=1}^{\infty} \alpha_n |\psi_n\rangle$ converges in M if and only if $\sum_{n=1}^{\infty} |\alpha_n|^2$ converges in \mathbb{R} .

- From the last Theorem, we see that if $\{|m_1\rangle \dots |m_l\rangle \dots\}$ is an **orthonormal** Schauder \mathbb{T} -basis and

$$\sum_{l=1}^{\infty} (\mathbf{e}_1 z_{\hat{1}} + \mathbf{e}_2 z_{\hat{2}}) |m_l\rangle$$

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- In particular, $\sum_{l=1}^{\infty} |z_{l\hat{k}}|^2$ also converges. Hence $\sum_{l=1}^{\infty} z_{l\hat{k}} |m_l\rangle$ converges and this allows to define projectors P_1 and P_2 from M to V as

$$P_k(|\psi\rangle) := \sum_{l=1}^{\infty} z_{l\hat{k}} |m_l\rangle, \quad k = 1, 2.$$

- Therefore, any $|\psi\rangle \in M$ can be decomposed uniquely as

$$|\psi\rangle = e_1 P_1(|\psi\rangle) + e_2 P_2(|\psi\rangle).$$

- As in the finite-dimensional case, one can easily show that ket projectors and idempotent-basis projectors (denoted with the same symbol) satisfy the following, for $k = 1, 2$:

$$P_k(s|\psi\rangle + t|\phi\rangle) = P_k(s) P_k(|\psi\rangle) + P_k(t) P_k(|\phi\rangle).$$

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- Complex Hilbert spaces are fundamental tools of quantum mechanics. We should therefore expect that bicomplex Hilbert spaces should be relevant to any attempted generalization of quantum mechanics to bicomplex numbers. Let us examine the example of the quantum harmonic oscillator.
- We start with the following function space. Let n be a nonnegative integer and let α be a positive real number. Consider the following function of a real variable x :

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- Let S be the set of all finite linear combinations of functions $f_{n,\alpha}(x)$, with complex coefficients. Furthermore, let a bicomplex function $u(x)$ be defined as

$$u(x) = \mathbf{e}_1 u_{\hat{1}}(x) + \mathbf{e}_2 u_{\hat{2}}(x),$$

where $u_{\hat{1}}$ and $u_{\hat{2}}$ are both in S . The set of all functions $u(x)$ is a \mathbb{T} -module, denoted by M_S .

- Let $u(x)$ and $v(x)$ both belong to M_S . We define a mapping (u, v) of this pair of functions into \mathbb{D}^+ as follows:

$$(u, v) := \int_{-\infty}^{\infty} u^{\dagger 3}(x)v(x)dx = \int_{-\infty}^{\infty} [\mathbf{e}_1 \bar{u}_{\hat{1}}(x)v_{\hat{1}}(x) + \mathbf{e}_2 \bar{u}_{\hat{2}}(x)v_{\hat{2}}(x)] dx.$$

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- It is not hard to see that the last equation is always finite and satisfies all the properties of a bicomplex scalar product.
- Let $\xi = \mathbf{e}_1 \xi_1 + \mathbf{e}_2 \xi_2$ be in \mathbb{D}^+ and let us define two operators X and P that act on elements of M_S as follows:

$$X\{u(x)\} := xu(x), \quad P\{u(x)\} := -\mathbf{i}_1 \hbar \xi \frac{du(x)}{dx}.$$

- It is not difficult to show that

$$[X, P] = \mathbf{i}_1 \hbar \xi I.$$

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$$X\{u(x)\} := xu(x), \quad P\{u(x)\} := -\mathbf{i}_1 \hbar \xi \frac{du(x)}{dx}.$$

- It is not difficult to show that

$$[X, P] = \mathbf{i}_1 \hbar \xi I.$$

- Note that the action of X and P on elements of M_S always yields elements of M_S . That is, X and P are defined on all of M_S .

- Let m and ω be two positive real numbers. We define the bicomplex harmonic oscillator Hamiltonian as follows:

$$H := \frac{1}{2m}P^2 + \frac{1}{2}m\omega^2X^2.$$

- The problem of the bicomplex quantum harmonic oscillator consists in finding the eigenvalues and eigenfunctions of H .
- That problem was solved in a previous paper on the topic. The results can be summarized as follows. Let θ_k ($k = \widehat{1}, \widehat{2}$) be defined as

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- Bicomplex harmonic oscillator eigenfunctions can then be written as (the most general eigenfunction would have different l indices in the two terms):

$$\begin{aligned}\phi_l(x) &= \mathbf{e}_1 \phi_{l\hat{1}} + \mathbf{e}_2 \phi_{l\hat{2}} \\ &= \mathbf{e}_1 \left[\sqrt{\frac{m\omega}{\pi\hbar\xi_1}} \frac{1}{2^l l!} \right]^{1/2} e^{-\theta_1^2/2} H_l(\theta_1) + \mathbf{e}_1 \left[\sqrt{\frac{m\omega}{\pi\hbar\xi_2}} \frac{1}{2^l l!} \right]^{1/2} e^{-\theta_2^2/2} H_l(\theta_2),\end{aligned}$$

where H_l are Hermite polynomials. The last equation can be written more succinctly as

$$\phi_l(x) = \left[\sqrt{\frac{m\omega}{\pi\hbar\xi}} \frac{1}{2^l l!} \right]^{1/2} e^{-\theta^2/2} H_l(\theta),$$

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$$\theta := \mathbf{e}_1 \theta_{\hat{1}} + \mathbf{e}_2 \theta_{\hat{2}} \quad \text{and} \quad H_l(\theta) := \mathbf{e}_1 H_l(\theta_{\hat{1}}) + \mathbf{e}_2 H_l(\theta_{\hat{2}}).$$

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- Finally, we can show that the collection of all finite linear combinations of bicomplex functions $\phi_I(x)$, with bicomplex coefficients, is a \mathbb{T} -module. Specifically,

$$\tilde{M} := \left\{ \sum_I w_I \phi_I(x) \mid w_I \in \mathbb{T} \right\}.$$

- Since \tilde{M} only involves finite linear combinations of the functions ϕ_I , it is not complete. With the methods developed in this paper, however, we can extend \tilde{M} to a complete module, in fact to a bicomplex Hilbert space.

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- We can define two vector spaces \tilde{V}_1 and \tilde{V}_2 as $\tilde{V}_1 = \mathbf{e}_1\tilde{M}$ and $\tilde{V}_2 = \mathbf{e}_2\tilde{M}$. It is clear that \tilde{V}_1 contains all functions $\mathbf{e}_1\phi_{\hat{1}}$ and \tilde{V}_2 contains all $\mathbf{e}_2\phi_{\hat{2}}$. Now the functions $\phi_{\hat{1}}$ and $\phi_{\hat{2}}$ are normalized eigenfunctions of the usual quantum harmonic oscillator (with \hbar replaced by $\hbar\xi_{\hat{1}}$ or $\hbar\xi_{\hat{2}}$). It is well-known that, as a Schauder basis, these eigenfunctions generate $L^2(\mathbb{R})$.
- Let $u(x)$ be defined as before, except that $u_{\hat{1}}(x)$ and $u_{\hat{2}}(x)$ are both taken as $L^2(\mathbb{R})$ functions. Clearly, the set of all $u(x)$ makes up a \mathbb{T} -module, which we shall denote by M . With the scalar product, M becomes a bicomplex pre-Hilbert space. Since $L^2(\mathbb{R})$ is complete we obtain:

Corollary

M is a bicomplex Hilbert space.

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