





Bicomplex Numbers

In 1892, in search for and development of special algebras, Corrado Segre (1860-1924) published a paper in which he treated an infinite set of algebras whose elements he called bicomplex numbers, tricomplex numbers,..., n-complex numbers.

We define **bicomplex numbers** as follows :

$\mathbb{C}_2 := \{a + bi_1 + ci_2 + dj : i_1{}^2 = i_2{}^2 = -1, \ j^2 $	1}
where $i_2 j = j i_2 = -i_1$, $i_1 j = j i_1 = -i_2$, $i_2 i_1 = i_1 i_2$	$_{2} = j$
and $a, b, c, d \in \mathbb{R}$.	

We remark that we can write a bicomplex number $a + bi_1 + ci_2 + dj$ as :

$$((a+bi_1) + (c+di_1)i_2 = z_1 + z_2i_2)$$

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where $z_1, z_2 \in \mathbb{C}_1 := \{x + yi_1 : i_1^2 = -1\}$. Thus, \mathbb{C}_2 can be viewed as the *complexification* of \mathbb{C}_1 and a bicomplex number can be seen as an element of \mathbb{C}^2 . Moreover, \mathbb{C}_2 is a **commutative unitary ring** with the following characterization for the non-invertible elements.

Let $w = z_1 + z_2 i_2 \in \mathbb{C}_2$. Then w is **noninvertible** if and only if :

$$z_1^2 + z_2^2 = 0.$$

Bicomplex Analysis

It is also possible to define **differentiability** of a function at a point of \mathbb{C}_2 :

Definition 1 Let U be an open set of \mathbb{C}_2 and $w_0 \in U$. Then, $f: U \subseteq \mathbb{C}_2 \longrightarrow \mathbb{C}_2$ is said to be \mathbb{C}_2 -differentiable at w_0 with derivative equal to $f'(w_0) \in \mathbb{C}_2$ if

$$\lim_{\substack{w \to w_0 \\ (w - w_0 \ inv.)}} \frac{f(w) - f(w_0)}{w - w_0} = f'(w_0).$$

We will also say that the function f is \mathbb{C}_2 -holomorphic on an open set U iff f is \mathbb{C}_2 -differentiable at each point of U.

As we saw, a bicomplex number can be seen as an element of \mathbb{C}^2 , so a function $f(z_1 + z_2i_2) = f_1(z_1, z_2) + f_2(z_1, z_2)i_2$ of \mathbb{C}_2 can be seen as a mapping $f(z_1, z_2) = (f_1(z_1, z_2), f_2(z_1, z_2))$ of \mathbb{C}^2 . Here we have a characterization of such mappings :

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Theorem 1 Let U be an open set and $f : U \subseteq \mathbb{C}_2 \longrightarrow \mathbb{C}_2$. Let also $f(z_1 + z_2i_2) = f_1(z_1, z_2) + f_2(z_1, z_2)i_2$. Then f is \mathbb{C}_2 -holomorphic on U iff:

 f_1 and f_2 are holomorphic in z_1 and z_2

and,

$$\frac{\partial f_1}{\partial z_1} = \frac{\partial f_2}{\partial z_2}$$
 and $\frac{\partial f_2}{\partial z_1} = -\frac{\partial f_1}{\partial z_2}$ on U.

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Now, it is natural to define for \mathbb{C}^2 the following class of mappings :

Definition 2 The class of \mathbb{T} -holomorphic mappings on a open set $U \subseteq \mathbb{C}^2$ is defined as follows :

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$$TH(U) := \{ f : U \subseteq \mathbb{C}^2 \longrightarrow \mathbb{C}^2 | f \in H(U) \text{ and} \\ \frac{\partial f_1}{\partial z_1} = \frac{\partial f_2}{\partial z_2}, \ \frac{\partial f_2}{\partial z_1} = -\frac{\partial f_1}{\partial z_2} \text{ on } U \}.$$

It is the subclass of holomorphic mappings of \mathbb{C}^2 satisfying the *complexified Cauchy-Riemann* equations.

The idempotent basis

We remark that $f \in TH(U)$ iff f is \mathbb{C}_2 -holomorphic on U. It is also important to know that every bicomplex number $z_1 + z_2i_2$ has the following unique idempotent representation :

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$$z_1 + z_2 i_2 = (z_1 - z_2 i_1)e_1 + (z_1 + z_2 i_1)e_2$$

where $e_1 = \frac{1+j}{2}$ and $e_2 = \frac{1-j}{2}$.

This representation is very useful because : addition, multiplication and division can be done term-by-term. Also, an element will be noninvertible iff $z_1 - z_2i_1 = 0$ or $z_1 + z_2i_1 = 0$. The notion of **holomorphicity** can also be seen with this kind of notation. For this we need the following definition :

Definition 3 We say that $X \subseteq \mathbb{C}_2$ is a \mathbb{C}_2 -cartesian set determined by X_1 and X_2 if

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$$X = X_1 \times_e X_2 := \{ z_1 + z_2 i_2 \in \mathbb{C}_2 : z_1 + z_2 i_2 = 0 \}$$

 $w_1e_1 + w_2e_2, (w_1, w_2) \in X_1 \times X_2\}.$

Remark :

If X_1 and X_2 are domains of \mathbb{C}_1 then $X_1 \times_e X_2$ is also a domain of \mathbb{C}_2 .

Now, it is possible to state the following striking theorems :

Theorem 2 If $f_{e1} : X_1 \longrightarrow \mathbb{C}_1$ and $f_{e2} : X_1 \longrightarrow \mathbb{C}_1$ are holomorphic functions of \mathbb{C}_1 on the domains X_1 and X_2 respectively, then the function $f : X_1 \times_e X_2 \longrightarrow \mathbb{C}_2$ defined as

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$$f(z_1 + z_2 i_2) = f_{e1}(z_1 - z_2 i_1)e_1 + f_{e2}(z_1 + z_2 i_1)e_2,$$

 $\forall z_1 + z_2 i_2 \in X_1 \times_e X_2$ is $(\mathbb{T}-holomorphic)$ on the domain $X_1 \times_e X_2$.



Such a disc is called a univalent disc for f. The Bloch constant may be described as :

$$\beta = \inf\{\beta_f : f \in H(B) \text{ with } f'(0) = 1\}$$

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where

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 $\beta_f = \sup\{b : f(B) \text{ contains a univalent disc of radius b}\}.$

The following upper and lower estimates for β were found by L. Ahlfors and H. Grunsky :

$$0.43\dots = \frac{\sqrt{3}}{4} \le \beta \le \frac{\Gamma(1/3)\Gamma(11/12)}{\Gamma(1/4)(1+\sqrt{3})^{1/2}} = 0.47\dots.$$

It is conjectured that the correct value of β is precisely this upper bound.

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Passing to several complex variables, a holomorphic mapping f from a domain in \mathbb{C}^n into \mathbb{C}^n is said to be nondegenerate if $det\mathcal{J}_f$ is not identically zero on the domain. Let B^n denote the open unit ball in \mathbb{C}^n . A nondegenerate mapping f from B^n into \mathbb{C}^n is said to be normalized if $det\mathcal{J}_f(0) = 1$, where 0 denotes the origin in \mathbb{C}^n . For such f we denote by β_f the supremum of values b such that the image $f(B^n)$ contains a univalent ball of radius b.

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If we fix K > 0 and consider the holomorphic mapping $f: \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ defined by

$$f(z_1, z_2) = (z_1/\sqrt{K}, \sqrt{K}z_2),$$

then, f is normalized but $\beta_f = 1/\sqrt{K}$. Since K can be chosen arbitrarily large, we see that there is no Bloch theorem for general holomorphic mappings, when n > 1.

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One might argue that the correct generalization of the normalization "f'(0) = 1" to several variables is not " $det \mathcal{J}_f(0) = 1$ ". However, there are also examples of holomorphic mappings f, with the stronger normalization $\mathcal{J}_f(0) = I$ (I is the identity mapping) and for which β_f is arbitrarily small.

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Thus, we see that for n > 1 we need to restrict the class of mappings to a more specific subclass to obtain a Bloch theorem. One of the well known subclasses is the class of K-quasiregular mappings. For such a class of mappings, it is possible to find a Bloch theorem in \mathbb{C}^n . The next theorem affirms that there exists also a Bloch constant for the class of T-holomorphic mappings on the unit ball of \mathbb{C}^2 .

Bloch Theorem in \mathbb{C}_2

Theorem 4 There is a positive constant d such that if $f \in TH(B^2)$ with $\mathcal{J}_f(0) = I$, then f maps some subdomain of B^2 biholomorphically onto a ball of radius d.

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We use the following notations :

Notation 1

 $\delta := \inf\{\delta_f : f \in TH(B^2(0,1)) \text{ with } \mathcal{J}_f(0) = I\},\$

where $\delta_f := \sup\{d : f(B^2(0,1)) \text{ contains a univalent} ball of radius d\}.$

We find the following estimates for our Bloch constant δ on the unit ball :

Theorem 5

$$\frac{\beta}{\sqrt{2}} \le \delta \le \sqrt{2}\beta,$$

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where β is the Bloch constant of one variable.

Remark :

If we replace the balls of \mathbb{C}^2 by the bicomplex "discs" $D(0,r) := B^1(0,r) \times_e B^1(0,r)$ where $B^n(0,r)$ is the open ball of $\mathbb{C}_1^n \simeq \mathbb{C}^n$, the Bloch constant has the same value as the classical Bloch constant for one variable.

Hyperholomorphy vs Quasiregularity

Now, to justify our Bloch theorem on the unit ball, we need to prove that the new class of mappings is not totally included in the class of K-quasiregular mappings.

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It is easy to show the following characterization :

Theorem 6 If $f \in TH(B^2)$ then f is K-quasiregular iff

$$\left|\frac{\partial f_1}{\partial z_1}\right|^2 + \left|\frac{\partial f_2}{\partial z_1}\right|^2 \le K^2 \left|\left(\frac{\partial f_1}{\partial z_1}\right)^2 + \left(\frac{\partial f_2}{\partial z_1}\right)^2\right| \text{ on } B^2.$$



Example 2 If $f(w) = w + \frac{w^2}{2}$, then f is an entire \mathbb{T} holomorphic (normalized) mapping, but for all K is not K-quasiregular. Proof. The function f is normalized because f'(w) = 1+wand then f'(0) = 1. Also, $w_0 = -1/2 + 1/2j$ is in B^2

Slide 21 Proof. The function f is normalized because f'(w) = 1+wand then f'(0) = 1. Also, $w_0 = -1/2 + 1/2j$ is in B^2 with $f'(w_0) = 1/2 + 1/2j$ which is noninvertible. Hence, f cannot satisfy the criteria of Theorem 6 at w_0 and then for all K, f cannot be K-quasiregular. \Box

Picard Theorem in \mathbb{C}_2

The Picard theorem follows from Bloch's theorem in one variable. It is then interesting to ask whether the same is possible in the case of \mathbb{T} -holomorphic mappings in \mathbb{C}^2 . However, here we can directly find a Picard theorem without invoking our Bloch theorem :

Theorem 7 (Picard) Let $f \in TH(\mathbb{C}^2)$. If there are two bicomplex numbers α, β such that $\alpha - \beta$ is invertible and for which the set $\{w \in \mathbb{C}_2 : w - \alpha \text{ is noninvertible}\}$

 $\cup \{ w \in \mathbb{C}_2 : w - \beta \text{ is noninvertible} \}$

is not in the range of f, then f is constant.



Theorem 8 (Casorati-Weierstrass) Let $f \in TH(\mathbb{C}^2)$ with f'(w) not identically noninvertible. Then, $f(\mathbb{C}^2)$ is dense in \mathbb{C}^2 .

A famous example of Fatou and Bieberbach shows that the usual formulation of the Picard theorem in \mathbb{C} does not extend to holomorphic mappings in \mathbb{C}^2 .

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In this connection, we have some interesting consequences of Theorem 8 which can be interpreted as an other kind of little Picard theorem for bicomplex numbers :

Corollary 1 There is no nondegenerate \mathbb{T} -holomorphic mapping

 $f: \mathbb{C}^2 \longrightarrow \mathbb{C}^2$

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such that $\mathbb{C}^2 \setminus f(\mathbb{C}^2)$ contains a ball.

Corollary 2 Fatou-Bieberbach examples cannot be \mathbb{T} -holomorphic mappings, i.e. they cannot satisfy the complexified Cauchy-Riemann equations.