Outline

- Bicomplex Numbers
- Bicomplex Dynamics
  - Bicomplex Mandelbrot Set
  - Bicomplex Filled-Julia Sets
  - Definition of the Tetrabrot
- Generalized Fatou-Julia Theorem
  - Strong Basin of Attraction of $\infty$
  - Cantor Sets in $\mathbb{R}^4$
- Bicomplex Filled-Julia Sets in $\mathbb{R}^3$
Quaternionic Dynamics

In 1982, A. Norton gave some straightforward algorithms for the generation and display in 3-D of fractal shapes. For the first time, iteration with quaternions appeared. Subsequently, theoretical results have been treated for the quaternionic Mandelbrot set defined by the quadratic polynomial in the quaternions of the form $q^2 + c$. However, S. Bedding and K. Briggs have established that there is no interesting dynamics for this approach and it does not play any fundamental role analogous to that for the map $z^2 + c$ in the complex plane.

Bicomplex Numbers

In 1892, in search for and development of special algebras, Corrado Segre (1860-1924) published a paper in which he treated an infinite set of algebras whose elements he called bicomplex numbers, tricomplex numbers, ..., n-complex numbers.

We define bicomplex numbers as follows:

$$\mathbb{T} := \{a + bi_1 + ci_2 + dj : i_1^2 = i_2^2 = -1, j^2 = 1\}$$

where $i_2j = ji_2 = -i_1$, $i_1j = ji_1 = -i_2$, $i_2i_1 = i_1i_2 = j$

and $a, b, c, d \in \mathbb{R}$.
We remark that we can write a bicomplex number \( a + bi_1 + ci_2 + dj \) as:
\[
(a + bi_1) + (c + di_1)i_2 = z_1 + z_2i_2
\]
where \( z_1, z_2 \in \mathbb{C}(i_1) := \{x+yi_1 : i_1^2 = -1\} \). Thus, \( \mathbb{T} \) can be viewed as the complexification of \( \mathbb{C}(i_1) \) and a bicomplex number can be seen as an element of \( \mathbb{T} \). Moreover, \( \mathbb{T} \) is a **commutative unitary ring** with the following characterization for the noninvertible elements.

Let \( w = z_1 + z_2i_2 \in \mathbb{T} \). Then \( w \) is **noninvertible** if and only if:
\[
z_1^2 + z_2^2 = 0.
\]

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The idempotent basis

It is also important to know that every bicomplex number \( z_1 + z_2i_2 \) has the following unique idempotent representation:
\[
z_1 + z_2i_2 = (z_1 - z_2i_1)e_1 + (z_1 + z_2i_1)e_2
\]
where \( e_1 = \frac{1+i}{2} \) and \( e_2 = \frac{1-i}{2} \).

This representation is very useful because: addition, multiplication and division can be done term-by-term. Also, an element will be non-invertible iff \( z_1 - z_2i_1 = 0 \) or \( z_1 + z_2i_1 = 0 \).
From the idempotent basis, it is now possible to define the notion of the **bicomplex cartesian product**.

**Definition 1** We say that \( X \subseteq \mathbb{T} \) is a bicomplex cartesian set determined by \( X_1 \) and \( X_2 \) if

\[
X = X_1 \times_c X_2 := \{ z_1 + z_2 i_2 \in \mathbb{T} : z_1 + z_2 i_2 = w_1 e_1 + w_2 e_2, (w_1, w_2) \in X_1 \times X_2 \}.
\]

Now, let us define a version of the Mandelbrot set for the bicomplex numbers:

**Definition 2** Let \( P_c(w) = w^2 + c \) where \( w, c \in \mathbb{T} \) and \( P_c^n(w) := (P_c^{n-1} \circ P_c)(w) \). Then the generalized Mandelbrot set for bicomplex numbers is defined as follows:

\[
\mathcal{M}_2 = \{ c \in \mathbb{T} : P_c^n(0) \nrightarrow \infty \}.
\]

From this definition we obtain the following result:

**Theorem 1** The generalized Mandelbrot set \( \mathcal{M}_2 \) is connected.
Generalized Filled-Julia Sets

It is also possible to generalize the notion of filled-Julia set for the bicomplex numbers:

**Definition 3** The generalized filled-Julia set for bicomplex numbers is defined as follows: \((c \in \mathbb{T})\)

\[ K_{2,c} = \{ w \in \mathbb{T} : P_c^{\circ n}(w) \not\to \infty \} \]

Finally, we obtain this relationship between the generalized Mandelbrot set and the generalized filled-Julia sets for the bicomplex numbers:

**Theorem 2** \( c \in \mathcal{M}_2 \iff K_{2,c} \text{ is connected} \).

Dynamics of Several Complex Variables

The polynomial \( P_c(w) = w^2 + c \) is the following mapping of \( \mathbb{C}^2 \) : \(( z_1^2 - z_2^2 + c_1, 2z_1z_2 + c_2 )\) where \( w = z_1 + z_2i_2 \) and \( c = c_1 + c_2i_2 \). We note that this mapping is not a holomorphic automorphism of \( \mathbb{C}^2 \).
Previously, we established a version of the Mandelbrot set in dimension four. We are able now to give a version of the Mandelbrot set in dimension three using the definition for $M_2$. The idea is to preserve the Mandelbrot set inside $M_2$. Then, if we restrict the algorithm to the points of the form $a + bi_1 + ci_2$ where $a, b, c \in \mathbb{R}$, we preserve the Mandelbrot set on two perpendicular complex planes and we stay in $\mathbb{R}^3$. This is the first argument to justify the following definition.

**The Tetrabrot**

**Definition 4** The "Tetrabrot" is defined as follows:

$T = \{ a + bi_1 + ci_2 + dj \in T : d = 0 \text{ and } P_e^{\infty}(0) \not\to \infty \}$.

It is possible to compute the infinite divergence layers of the Tetrabrot. We have to note at this step that each divergence layer will hide the others. For example, Fig. 2 is an illustration for the Tetrabrot of one of its divergence layers in correspondence with the divergence layer illustrated in Fig. 1(A) for the Mandelbrot set.
In fact, the Tetrabrot is inside Fig. 2. It is possible to see a part of the Tetrabrot (see Fig. 3) if we cut a piece of Fig. 2. In Fig. 3, the colors represent the other divergence layers. It is also possible to compute other divergence layers (see Figs. 4, 5, 6 and 7). Figure 7 begins to be close to the set we wish to approach; then Fig. 7 with its cut plane gives certainly a good idea of the Tetrabrot.

Moreover, we observe that the specific enlargement of Fig. 7 between A and B (Fig. 24) confirms that the Tetrabrot could be disconnected.

Finally, to define the Tetrabrot we have put the last coordinate “j” equal to zero. In fact it is possible to do the same if we fix the last coordinate equal to a number different from zero. However, if we do that, we lose the beautiful symmetry of the Tetrabrot. Figure 8 gives an illustration of this phenomenon for a fixed “dj” with $d \neq 0$. 
Let us recall the definition of a Cantor set in $\mathbb{R}^n$:

**Definition 5** A Cantor set is defined as a compact, perfect, totally disconnected subset in $\mathbb{R}^n$.

**Remark 1** Any such set is homeomorphic to the Cantor middle third set and therefore deserves the name of Cantor set.

The following classical theorem establishes a connection between the Cantor sets and the filled-Julia sets:

**Theorem 3 (P. Fatou and G. Julia)** Let $\mathcal{K}_c$ be a filled-Julia set of the family of complex quadratic polynomials $P_c(z) = z^2 + c$ in the complex plane, and $\mathcal{A}_c(\infty) = \mathbb{C} \setminus \mathcal{K}_c$ the basin of attraction of $\infty$ for $P_c(z) = z^2 + c$. Then

1. $0 \in \mathcal{K}_c \iff \mathcal{K}_c$ is connected;

2. $0 \in \mathcal{A}_c(\infty) \iff \mathcal{K}_c$ is a Cantor set.
Now, let us define the concept of basin of attraction of $\infty$ in the context of bicomplex numbers:

**Definition 6** Let $\mathcal{K}_{2,c}$ be a filled-Julia set of the family of bicomplex quadratic polynomials $P_c(w) = w^2 + c$ in $\mathbb{T}$. We define $A_{2,c}(\infty) = \mathbb{T} \setminus \mathcal{K}_{2,c}$ as the basin of attraction of $\infty$ for $P_c(w) = w^2 + c$. We note that

$$A_{2,c}(\infty) = \{w \in \mathbb{T} \mid P_{c}^{\circ n}(w) \to \infty\}.$$ 

The next definition will be well justified in regard of the theorem below.

**Definition 7** We define

$$SA_{2,c}(\infty) = (\mathcal{K}_{c_1-c_2i_1})^c \times_e (\mathcal{K}_{c_1+c_2i_1})^c$$

$$= A_{c_1-c_2i_1}(\infty) \times_e A_{c_1+c_2i_1}(\infty)$$

as the **strong basin of attraction of** $\infty$ for $P_c(w) = w^2 + c$ where $c = (c_1-c_2i_1)e_1 + (c_1+c_2i_1)e_2$. We note that

$$SA_{2,c}(\infty) \subset A_{2,c}(\infty).$$
The following theorem gives a characterization of the filled-Julia sets for bicomplex numbers and introduces naturally the idea of Cantor sets in $\mathbb{R}^4$.

**Theorem 4 (Fatou-Julia Theorem in $T$)** Let $K_{2,c}$ be a filled-Julia sets for bicomplex numbers and $c \in T$. Then

(1) $0 \in K_{2,c} \iff K_{2,c}$ is connected;

(2) $0 \in SA_{2,c}(\infty) \iff K_{2,c}$ is a Cantor set in $T$;

(3) $0 \in A_{2,c}(\infty) \setminus SA_{2,c}(\infty) \iff K_{2,c}$ is disconnected but not totally disconnected.

Now, it is possible to illustrate by figures in $\mathbb{R}^3$ the connections between the various cases of the last theorem when the filled-Julia sets come from points around or inside the Tetrabrot. In fact, we are in the case 1 of the last theorem if and only if the filled-Julia sets come from points inside the Tetrabrot. The other cases are established from points inside the infinite divergence layers of the Tetrabrot. Fig. T1 and T2 are an illustration of this phenomenon where the red zones are the points $c$ which satisfy the case 3 and the other colors the points $c$ which satisfy the case 2.
More specifically, the Fig. T2 makes it possible to observe this phenomenon on the bottom inside of the Tetrabrot for a specific divergence layer; the colors on the cut plane are an illustration of the other divergence layers of the Tetrabrot. We note also that the colors for the case 2 have been computed from the average of each divergence layer obtained from the number of iterations needed to know that zero is inside \( A_{c_1 - c_2 i_1}(\infty) \) and \( A_{c_1 + c_2 i_1}(\infty) \).

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**The Filled-Julia sets in \( \mathbb{R}^3 \)**

The same process as for the Tetrabrot yields a version of the filled-Julia sets in \( \mathbb{R}^3 \). The next definition defined the bicomplex filled-Julia sets in \( \mathbb{R}^3 \).

**Definition 8** A bicomplex filled-Julia set in \( \mathbb{R}^3 \) is defined as follows: \( (c \in \mathbb{T}) \)

\[
\mathcal{L}_{2,c} = \{ w = a + bi_1 + ci_2 + dj \in \mathbb{T} : d = 0 \text{ and } P_c^\infty(w) \not\rightarrow \infty \}.
\]
Figure 17 is an illustration of the filled-Julia set for the Tetrabrot at the same point $c = 0.25$ as the filled-Julia set $D$ of Fig. 1. Hence, Fig. 17 is a kind of generalization of the filled-Julia set $K_{0.25}$ in the complex plane. In the same manner, Figs. 20-23 are an illustration of the filled-Julia set at $c = i_1$ for different divergence layers to infinity. We remark that Fig. 23 is a good approximation of this set and an interesting generalization of Fig. 1(E).

In their book “Complex Dynamics”, L. Carleson and T. W. Gamelin have remarked this interesting fact: “One striking feature of $\mathcal{M}$ is that shapes of certain of the Julia sets $\mathcal{J}_c$ in dynamic space ($z$-space) are reflected in the shape of $\mathcal{M}$ near the corresponding points in parameter space ($c$-shape).”. For the Tetrabrot, we obtain similar result. For example, Fig. 18 is the Fig. 17 with the same kind of cut as for the Tetrabrot in Fig. 7. Hence, we see that inside the shape in Fig. 17, we have the same shape as inside the Tetrabrot near the point $c = 0.25$. 
This phenomenon has also been illustrated in Fig. 19 where we have put together the border of the Tetrabrot (the enlargement of Fig. 7(B)) and the associated “filled-Julia” set at the point $c = -1.16 - 0.25i_1$ along the border. We see clearly that this “filled-Julia” set imitates the border of the Tetrabrot.

The same process can also be used when the filled-Julia sets are not connected. In that case we established these following results:

**Lemma 1** Let $c \in \mathbb{T}$ and $\mathcal{K}_{2,c}$ a Cantor set in $\mathbb{R}^4$. Then $\mathcal{L}_{2,c}$ is compact and totally disconnected in $\mathbb{R}^3$.

**Theorem 5** Let $c \in \mathbb{C}(i_1) := \{x + yi_1 : i_1^2 = -1\}$ and $\mathcal{K}_c$ a Cantor set. Then, $\mathcal{K}_{2,c}$ is a Cantor set in $\mathbb{R}^4$ and $\mathcal{L}_{2,c}$ is the union of a Cantor set in $\mathbb{R}^3$ and a set which is at most countable.
Figures C1-C6, C7-C12 and C13-C17 give an illustration of such “Cantor sets” in $\mathbb{R}^3$. In fact, Figures C1-C6 and C7-C12 are respectively close to the filled-Julia sets of Fig. 17 and 20 without coming from points inside the Tetrabrot. In each case, each step used a divergence layer to infinity closer to the true set. With this construction, we remark that we visually obtain a result similar to the geometric construction of the Cantor middle third set. Moreover, Fig. C18 is an illustration of $L_{2,c}$ for a filled-Julia set $\mathcal{K}_{2,c}$ where $0 \in A_{2,c}(\infty) \setminus S A_{2,c}(\infty)$. We note that this set in $\mathbb{R}^3$ has connected components.

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