On a relation of bicomplex pseudoanalytic function theory to the complexified stationary Schrödinger equation

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Abstract

Using three different representations of the bicomplex numbers $\mathbb{T} \cong \mathrm{Cl}_{\mathbb{C}}(1,0) \cong \mathrm{Cl}_{\mathbb{C}}(0,1)$, which is a commutative ring with zero divisors defined by $\mathbb{T} = \{w_0 + w_1\mathbf{i_1} + w_2\mathbf{i_2} + w_3\mathbf{j} \mid w_0, w_1, w_2, w_3 \in \mathbb{R}\}$ where $\mathbf{i_1^2} = -1$, $\mathbf{i_2^2} = -1$, $\mathbf{j}^2 = 1$ and $\mathbf{i_1}\mathbf{i_2} = \mathbf{j} = \mathbf{i_2}\mathbf{i_1}$, we construct three classes of bicomplex pseudoanalytic functions. In particular, we obtain some specific systems of Vekua equations of two complex variables and we established some connections between one of these systems and the classical Vekua equations. We consider also the complexification of the real stationary two-dimensional Schrödinger equation. With the aid of any of its particular solutions, we construct a specific bicomplex Vekua equation possessing the following special property. The scalar parts of its solutions are solutions of the original complexified Schrödinger equation and the vectorial parts are solutions of another complexified Schrödinger equation.

Keywords: Bicomplex Numbers, Hyperbolic Numbers, Complex Clifford Algebras, Pseudoanalytic Functions, Second Order Elliptic Operator, Two-dimensional Stationary Schrödinger Equation.

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1 Introduction

The pseudoanalytic function theory was independently developed by two prominent mathematicians, I.N. Vekua (see [27]) and L. Bers (see [1, 4, 5]). Historically, the theory became one of the important impulses for developing the general theory of elliptic systems. More recently, it has been established by V.V. Kravchenko (see [12, 14]) that with the aid of any particular solutions of the real stationary two-dimensional Schrödinger equation we can construct a Vekua equation possessing the following special property. The real parts of its solutions are solutions of the original Schrödinger equation and the imaginary parts are solution of an associated Schrödinger equation with a potential having the form of a potential obtained after a Darboux transform. Using Bers's theory of Taylor series for pseudoanalytic function, the author obtain a locally complete system of solutions of the original Schrödinger equation which can be constructed explicitly for an ample class of Schrödinger equation. Subsequently, V.V. Kravchenko (see [24]) gives a generalization of the factorization technics developed in [12] for the more general two-dimensional elliptic operator $(\text{div } p \text{ grad} + q)u = 0$, and consider the case where p and q are complex functions. In particular, the author had to consider a bicomplex Vekua equation of a special form to be able to obtain the following more general property. The scalar parts of the bicomplex Vekua equation considered are solutions of the original Schrödinger equation with a complex-valued potential. However, the case using the complex functions is more complicated and the author let as an open question the proof of expansion and convergence theorems for the bicomplex Vekua equation considered.

In this article, using three different representations of the bicomplex numbers (see, e.g. [7, 18, 19, 20, 22, 23, 24]), which is a commutative ring with zero divisors defined by $\mathbb{T} = \{w_0 + w_1\mathbf{i_1} + w_2\mathbf{i_2} + w_3\mathbf{j} \mid w_0, w_1, w_2, w_3 \in \mathbb{R}\}$ where $\mathbf{i_1}^2 = -1$, $\mathbf{i_2}^2 = -1$, $\mathbf{j}^2 = 1$ and $\mathbf{i_1}\mathbf{i_2} = \mathbf{j} = \mathbf{i_2}\mathbf{i_1}$, we construct three classes of bicomplex pseudoanalytic functions. For every class we obtain a special type of bicomplex Vekua equation of two complex variables, of which the one considered by A. Castaneda and V.V. Kravchenko (see [6, 13]) when the domain is restricted to the complex (in $\mathbf{i_2}$) plane. Moreover, we established some connections between one of these systems of bicomplex Vekua equation and the classical Vekua equations. We consider also the complexification of the real stationary two-dimensional Schrödinger equation :

$$(\triangle_{\mathbb{C}} - \nu(z_1, z_2))f = 0$$

where $\omega = z_1 + z_2\mathbf{i_2} \in \mathbb{T}$ with $z_1, z_2 \in \mathbb{C}(\mathbf{i_1})$ and $\triangle_{\mathbb{C}}$ is the complexified Laplacian operator i.e. $\triangle_{\mathbb{C}} = \partial_{z_1}^2 + \partial_{z_2}^2$. With the aid of any of its particular solutions f_0 and the bicomplex operators :

$$\partial_{\omega^{\dagger_2}} = \frac{1}{2}\left(\partial_{z_1} + \mathbf{i_2}\partial_{z_2}\right) \text{ and } \partial_{\omega} = \frac{1}{2}\left(\partial_{z_1} - \mathbf{i_2}\partial_{z_2}\right),$$

we obtain the following factorization of the complexified Schrödinger equation

$$(\triangle_{\mathbb{C}} - \nu)\varphi = 4\left(\partial_{\omega^{\dagger_2}} + \frac{\partial_\omega f_0}{f_0}C\right)\left(\partial_\omega - \frac{\partial_\omega f_0}{f_0}C\right)\varphi$$

where C denote the $\dagger_2$-bicomplex conjugation operator, and we consider a specific bicomplex Vekua equation

$$\left(\partial_{\omega^{\dagger_2}} - \frac{\partial_{\omega^{\dagger_2}} f_0}{f_0}C\right)W = 0$$

possessing the following special property. The scalar parts of its solutions are solutions of the original complexified Schrödinger equation and the vectorial parts are solutions of another complexified Schrödinger equation with the following potential $-\nu(z_1,z_2) + \frac{2|\bigtriangledown_{\mathbb{C}} f_0|^2_{\mathbf{i_1}}}{f_2^2}$ where $|\omega|^2_{\mathbf{i_1}} = \omega\omega^{\dagger_2}$ $\forall\omega \in \mathbb{T}$ and $\bigtriangledown_{\mathbb{C}}$ is the complexified gradient operator i.e. $\bigtriangledown_{\mathbb{C}} = \partial_{z_1} + \mathbf{i_2}\partial_{z_2}$.

Finally, from the fact that the complexified Schrödinger equation contains the stationary two-dimensional Schrödinger equation

$$(\triangle - \nu(x,p))f = 0$$

and the Klein-Gordon equation

$$(\square - \nu(x,q))f = 0,$$

we show that our factorization of the complexified Schrödinger equation is a generalization of the factorization obtained in [12] for the stationary two-dimensional Schrödinger equation and for the factorization obtained in [15] for the Klein-Gordon equation.

2 Preliminaries

2.1 Bicomplex Numbers

Bicomplex numbers are defined as

$$\mathbb{T} := \{z_1 + z_2\mathbf{i_2} \mid z_1, z_2 \in \mathbb{C}(\mathbf{i_1})\} \tag{2.1}$$

where the imaginary units $\mathbf{i_1}, \mathbf{i_2}$ and $\mathbf{j}$ are governed by the rules: $\mathbf{i_1}^2 = \mathbf{i_2}^2 = -1$, $\mathbf{j}^2 = 1$ and

$$\begin{array}{rcccr} \mathbf{i_1}\mathbf{i_2} & = & \mathbf{i_2}\mathbf{i_1} & = & \mathbf{j}, \\ \mathbf{i_1}\mathbf{j} & = & \mathbf{j}\mathbf{i_1} & = & -\mathbf{i_2}, \\ \mathbf{i_2}\mathbf{j} & = & \mathbf{j}\mathbf{i_2} & = & -\mathbf{i_1}. \end{array} \tag{2.2}$$

Note that we define $\mathbb{C}(\mathbf{i}_k) := \{x + y\mathbf{i}_k \mid \mathbf{i}_k^2 = -1 \text{ and } x, y \in \mathbb{R}\}$ for $k = 1, 2$. Hence, it is easy to see that the multiplication of two bicomplex numbers is commutative. In fact, the bicomplex numbers

$$\mathbb{T} \cong \mathrm{Cl}_{\mathbb{C}}(1,0) \cong \mathrm{Cl}_{\mathbb{C}}(0,1)$$

are *unique* among the complex Clifford algebras in that they are commutative but not division algebras. It is also convenient to write the set of bicomplex numbers as

$$\mathbb{T} := \{w_0 + w_1\mathbf{i_1} + w_2\mathbf{i_2} + w_3\mathbf{j} \mid w_0, w_1, w_2, w_3 \in \mathbb{R}\}. \tag{2.3}$$

In particular, in equation (2.1), if we put $z_1 = x$ and $z_2 = y\mathbf{i_1}$ with $x, y \in \mathbb{R}$, then we obtain the following subalgebra of hyperbolic numbers, also called duplex numbers (see, e.g. [22, 26]):

$$\mathbb{D} := \{x + y\mathbf{j} \mid \mathbf{j}^2 = 1,\ x, y \in \mathbb{R}\} \cong \mathrm{Cl}_{\mathbb{R}}(0,1).$$

Complex conjugation plays an important role both for algebraic and geometric properties of $\mathbb{C}$. For bicomplex numbers, there are three possible conjugations. Let $w \in \mathbb{T}$ and $z_1, z_2 \in \mathbb{C}(\mathbf{i_1})$ such that $w = z_1 + z_2\mathbf{i_2}$. Then we define the three conjugations as:

$$w^{\dagger_1} = (z_1 + z_2\mathbf{i_2})^{\dagger_1} := \overline{z}_1 + \overline{z}_2\mathbf{i_2}, \tag{2.4a}$$
$$w^{\dagger_2} = (z_1 + z_2\mathbf{i_2})^{\dagger_2} := z_1 - z_2\mathbf{i_2}, \tag{2.4b}$$
$$w^{\dagger_3} = (z_1 + z_2\mathbf{i_2})^{\dagger_3} := \overline{z}_1 - \overline{z}_2\mathbf{i_2}, \tag{2.4c}$$

where $\overline{z}_k$ is the standard complex conjugate of complex numbers $z_k \in \mathbb{C}(\mathbf{i_1})$. If we say that the bicomplex number $w = z_1 + z_2\mathbf{i_2} = w_0 + w_1\mathbf{i_1} + w_2\mathbf{i_2} + w_3\mathbf{j}$ has the "signature" $(++++)$, then the conjugations of type 1,2 or 3 of w have, respectively, the signatures $(+-+-)$, $(++--)$ and $(+--+)$. We can verify easily that the composition of the conjugates gives the four-dimensional abelian Klein group:

$\circ$	$\dagger_0$	$\dagger_1$	$\dagger_2$	$\dagger_3$
$\dagger_0$	$\dagger_0$	$\dagger_1$	$\dagger_2$	$\dagger_3$
$\dagger_1$	$\dagger_1$	$\dagger_0$	$\dagger_3$	$\dagger_2$
$\dagger_2$	$\dagger_2$	$\dagger_3$	$\dagger_0$	$\dagger_1$
$\dagger_3$	$\dagger_3$	$\dagger_2$	$\dagger_1$	$\dagger_0$

(2.5)

where $w^{\dagger_0} := w\ \forall w \in \mathbb{T}$.

The three kinds of conjugation all have some of the standard properties of conjugations, such as:

$$(s+t)^{\dagger_k} = s^{\dagger_k} + t^{\dagger_k}, \tag{2.6}$$
$$\left(s^{\dagger_k}\right)^{\dagger_k} = s, \tag{2.7}$$
$$(s \cdot t)^{\dagger_k} = s^{\dagger_k} \cdot t^{\dagger_k}, \tag{2.8}$$

for $s, t \in \mathbb{T}$ and $k = 0, 1, 2, 3$.

We know that the product of a standard complex number with its conjugate gives the square of the Euclidean metric in $\mathbb{R}^2$. The analogs of this, for bicomplex numbers, are the following. Let $z_1, z_2 \in \mathbb{C}(\mathbf{i_1})$ and $w = z_1 + z_2\mathbf{i_2} \in \mathbb{T}$, then we have that [22]:

$$|w|^2_{\mathbf{i_1}} := w \cdot w^{\dagger_2} = z_1^2 + z_2^2 \in \mathbb{C}(\mathbf{i_1}), \tag{2.9a}$$

$$|w|^2_{\mathbf{i_2}} := w \cdot w^{\dagger_1} = \left(|z_1|^2 - |z_2|^2\right) + 2\mathrm{Re}(z_1\overline{z}_2)\mathbf{i_2} \in \mathbb{C}(\mathbf{i_2}), \tag{2.9b}$$

$$|w|^2_{\mathbf{j}} := w \cdot w^{\dagger_3} = \left(|z_1|^2 + |z_2|^2\right) - 2\mathrm{Im}(z_1\overline{z}_2)\mathbf{j} \in \mathbb{D}, \tag{2.9c}$$

where the subscript of the square modulus refers to the subalgebra $\mathbb{C}(\mathbf{i_1}), \mathbb{C}(\mathbf{i_2})$ or $\mathbb{D}$ of $\mathbb{T}$ in which w is projected.

Note that for $z_1, z_2 \in \mathbb{C}(\mathbf{i_1})$ and $w = z_1 + z_2\mathbf{i_2} \in \mathbb{T}$, we can define the usual (Euclidean in $\mathbb{R}^4$) norm of w as $|w| = \sqrt{|z_1|^2 + |z_2|^2} = \sqrt{\mathrm{Re}(|w|^2_{\mathbf{j}})}$.

It is easy to verify that $w \cdot \dfrac{w^{\dagger_2}}{|w|^2_{\mathbf{i_1}}} = 1$. Hence, the inverse of w is given by

$$w^{-1} = \frac{w^{\dagger_2}}{|w|^2_{\mathbf{i_1}}}. \tag{2.10}$$

From this, we find that the set $\mathcal{NC}$ of zero divisors of $\mathbb{T}$, called the *null-cone*, is given by $\{z_1 + z_2\mathbf{i_2} \mid z_1^2 + z_2^2 = 0\}$, which can be rewritten as

$$\mathcal{NC} = \{z(\mathbf{i_1} \pm \mathbf{i_2}) |\ z \in \mathbb{C}(\mathbf{i_1})\}. \tag{2.11}$$

It is also possible to define differentiability of a function at a point of $\mathbb{T}$:

Definition 1 *Let U be an open set of $\mathbb{T}$ and $w_0 \in U$. Then, $f : U \subseteq \mathbb{T} \longrightarrow \mathbb{T}$ is said to be $\mathbb{T}$-differentiable at w_0 with derivative equal to $f'(w_0) \in \mathbb{T}$ if*

$$\lim_{\substack{w \to w_0 \\ (w - w_0 \ inv.)}} \frac{f(w) - f(w_0)}{w - w_0} = f'(w_0).$$

We also say that the function f is $\mathbb{T}$-holomorphic on an open set U if and only if f is $\mathbb{T}$-differentiable at each point of U.

As we saw, a bicomplex number can be seen as an element of $\mathbb{C}^2$, so a function $f(z_1 + z_2\mathbf{i_2}) = f_1(z_1, z_2) + f_2(z_1, z_2)\mathbf{i_2}$ of $\mathbb{T}$ can be seen as a mapping $f(z_1, z_2) = (f_1(z_1, z_2), f_2(z_1, z_2))$ of $\mathbb{C}^2$. Here we have a characterization of such mappings:

Theorem 1 *Let U be an open set and $f : U \subseteq \mathbb{T} \longrightarrow \mathbb{T}$ such that $f \in C^1(U)$. Let also $f(z_1 + z_2\mathbf{i_2}) = f_1(z_1, z_2) + f_2(z_1, z_2)\mathbf{i_2}$. Then f is $\mathbb{T}$-holomorphic on U if and only if:*

f_1 and f_2 are holomorphic in z_1 and z_2

and,

$$\frac{\partial f_1}{\partial z_1} = \frac{\partial f_2}{\partial z_2} \ \textit{and}\ \frac{\partial f_2}{\partial z_1} = -\frac{\partial f_1}{\partial z_2} \ \textit{on}\ U.$$

Moreover, $f' = \frac{\partial f_1}{\partial z_1} + \frac{\partial f_2}{\partial z_1}\mathbf{i_2}$ *and* $f'(w)$ *is invertible if and only if* $det\mathcal{J}_f(w) \neq 0$.

This theorem can be obtained from results in [18] and [21]. Moreover, by the Hartogs theorem [25], it is possible to show that "$f \in C^1(U)$" can be dropped from the hypotheses. Hence, it is natural to define the corresponding class of mappings for $\mathbb{C}^2$:

Definition 2 *The class of* $\mathbb{T}$*-holomorphic mappings on a open set* $U \subseteq \mathbb{C}^2$ *is defined as follows:*

$$TH(U) := \{f : U \subseteq \mathbb{C}^2 \longrightarrow \mathbb{C}^2 | f \in H(U) \ \textit{and}\ \frac{\partial f_1}{\partial z_1} = \frac{\partial f_2}{\partial z_2}, \frac{\partial f_2}{\partial z_1} = -\frac{\partial f_1}{\partial z_2} \ \textit{on}\ U\}.$$

It is the subclass of holomorphic mappings of $\mathbb{C}^2$ *satisfying the complexified Cauchy-Riemann equations.*

We remark that $f \in TH(U)$ in terms of $\mathbb{C}^2$ if and only if f is $\mathbb{T}$-differentiable on U. It is also important to know that every bicomplex number $z_1 + z_2\mathbf{i_2}$ has the following unique idempotent representation:

$$z_1 + z_2\mathbf{i_2} = (z_1 - z_2\mathbf{i_1})\mathbf{e_1} + (z_1 + z_2\mathbf{i_1})\mathbf{e_2}. \tag{2.12}$$

where $\mathbf{e_1} = \frac{1+\mathbf{j}}{2}$ and $\mathbf{e_2} = \frac{1-\mathbf{j}}{2}$.

This representation is very useful because: addition, multiplication and division can be done term-by-term. Also, an element will be non-invertible if and only if $z_1 - z_2\mathbf{i_1} = 0$ or $z_1 + z_2\mathbf{i_1} = 0$.

The notion of holomorphicity can also be seen with this kind of notation. For this we need to define the projections $P_1, P_2 : \mathbb{T} \longrightarrow \mathbb{C}(\mathbf{i_1})$ as $P_1(z_1+z_2\mathbf{i_2}) = z_1 - z_2\mathbf{i_1}$ and $P_2(z_1 + z_2\mathbf{i_2}) = z_1 + z_2\mathbf{i_1}$. Also, we need the following definition:

Definition 3 *We say that* $X \subseteq \mathbb{T}$ *is a* $\mathbb{T}$*-cartesian set determined by* X_1 *and* X_2 *if* $X = X_1 \times_e X_2 := \{z_1+z_2\mathbf{i_2} \in \mathbb{T} : z_1+z_2\mathbf{i_2} = w_1\mathbf{e_1}+w_2\mathbf{e_2}, (w_1, w_2) \in X_1 \times X_2\}$.

In [1] it is shown that if X_1 and X_2 are domains of $\mathbb{C}(\mathbf{i_1})$ then $X_1 \times_e X_2$ is also a domain of $\mathbb{T}$. Now, it is possible to state the following striking theorems [18]:

Theorem 2 *If* $f_{e1} : X_1 \longrightarrow \mathbb{C}(\mathbf{i_1})$ *and* $f_{e2} : X_2 \longrightarrow \mathbb{C}(\mathbf{i_1})$ *are holomorphic functions of* $\mathbb{C}(\mathbf{i_1})$ *on the domains* X_1 *and* X_2 *respectively, then the function* $f : X_1 \times_e X_2 \longrightarrow \mathbb{T}$ *defined as*

$$f(z_1 + z_2\mathbf{i_2}) = f_{e1}(z_1 - z_2\mathbf{i_1})\mathbf{e_1} + f_{e2}(z_1 + z_2\mathbf{i_1})\mathbf{e_2} \ \forall\ z_1 + z_2\mathbf{i_2} \in X_1 \times_e X_2$$

is $\mathbb{T}$*-holomorphic on the domain* $X_1 \times_e X_2$ *and*

$$f'(z_1 + z_2\mathbf{i_2}) = f'_{e1}(z_1 - z_2\mathbf{i_1})\mathbf{e_1} + f'_{e2}(z_1 + z_2\mathbf{i_1})\mathbf{e_2}$$

$\forall\ z_1 + z_2\mathbf{i_2} \in X_1 \times_e X_2$.

Theorem 3 *Let X be a domain in $\mathbb{T}$, and let $f : X \longrightarrow \mathbb{T}$ be a $\mathbb{T}$-holomorphic function on X. Then there exist holomorphic functions $f_{e1} : X_1 \longrightarrow \mathbb{C}(\mathbf{i_1})$ and $f_{e2} : X_2 \longrightarrow \mathbb{C}(\mathbf{i_1})$ with $X_1 = P_1(X)$ and $X_2 = P_2(X)$, such that:*

$$f(z_1 + z_2\mathbf{i_2}) = f_{e1}(z_1 - z_2\mathbf{i_1})\mathbf{e_1} + f_{e2}(z_1 + z_2\mathbf{i_1})\mathbf{e_2} \ \forall \ z_1 + z_2\mathbf{i_2} \in X.$$

We note here that X_1 and X_2 will also be domains of $\mathbb{C}(\mathbf{i_1})$.

3 Bicomplex Pseudoanalytic Functions

3.1 Elementary Bicomplex Derivative

We will first consider the variable $z = x + y\mathbf{i_1}$, where x and y are real variables and the corresponding formal differential operators

$$\partial_{\bar{z}} = \frac{1}{2}\left(\partial_x + \mathbf{i_1}\partial_y\right) \text{ and } \partial_z = \frac{1}{2}\left(\partial_x - \mathbf{i_1}\partial_y\right).$$

Notation $f_{\bar{z}}$ or f_z means the application of $\partial_{\bar{z}}$ or ∂_z respectively to a bicomplex function $f(z) = u(z) + v(z)\mathbf{i_1} + r(z)\mathbf{i_2} + s(z)\mathbf{j}$. The derivatives f_z, $f_{\bar{z}}$ "exist" if and only if f_x and f_y do. Note that

$$f_z = \frac{1}{2}\{(u_x + v_y) + (v_x - u_y)\mathbf{i_1} + (r_x + s_y)\mathbf{i_2} + (s_x - r_y)\mathbf{j})\}$$

and

$$f_{\bar{z}} = \frac{1}{2}\{(u_x - v_y) + (v_x + u_y)\mathbf{i_1} + (r_x - s_y)\mathbf{i_2} + (s_x + r_y)\mathbf{j})\}$$

In view of these operators,

$$f_{\bar{z}}(z) = 0 \Leftrightarrow \partial_{\bar{z}}[u(z) + v(z)\mathbf{i_1}] = 0 \text{ and } \partial_{\bar{z}}[r(z) + s(z)\mathbf{i_1}] = 0. \qquad (3.1)$$

i.e. $u_x = v_y, v_x = -u_y$ and $r_x = s_y, s_x = -r_y$ at $z \in \mathbb{C}(\mathbf{i_1})$.

We will now consider the bicomplex variable $\omega = z_1 + z_2\mathbf{i_2}$, where $z_1 = x_1 + y_1\mathbf{i_1}, z_2 = x_2 + y_2\mathbf{i_1} \in \mathbb{C}(\mathbf{i_1})$ and the corresponding formal differential operators

$$\partial_{\bar{\omega}} = \partial_{\omega^{\dagger_2}} = \frac{1}{2}\left(\partial_{z_1} + \mathbf{i_2}\partial_{z_2}\right), \quad \partial_{\omega} = \partial_{\omega^{\dagger_0}} = \frac{1}{2}\left(\partial_{z_1} - \mathbf{i_2}\partial_{z_2}\right)$$

$$\partial_{\omega^{\dagger_3}} = \frac{1}{2}\left(\partial_{\bar{z}_1} + \mathbf{i_2}\partial_{\bar{z}_2}\right) \quad \text{and} \quad \partial_{\omega^{\dagger_1}} = \frac{1}{2}\left(\partial_{\bar{z}_1} - \mathbf{i_2}\partial_{\bar{z}_2}\right).$$

Notation $f_{\omega^{\dagger_k}}$ for $k = 0, 1, 2, 3$ means the application of $f_{\omega^{\dagger_k}}$ respectively to a bicomplex function $f(\omega) = u(\omega) + v(\omega)\mathbf{i_1} + r(\omega)\mathbf{i_2} + s(\omega)\mathbf{j}$. The derivatives $f_{\omega^{\dagger_k}}$ "exist" for $k = 0, 1, 2, 3$ if and only if f_{x_l} and f_{y_l} exist for $l = 1, 2$. These bicomplex operators act on sums, products, etc. just as an ordinary derivative and we have the following result in the bicomplex function theory.

Lemma 1 *Let* $f(z_1+z_2\mathbf{i_2}) = f_1(z_1,z_2)+f_2(z_1,z_2)\mathbf{i_2} = u(z_1,z_2)+v(z_1,z_2)\mathbf{i_1}+r(z_1,z_2)\mathbf{i_2}+s(z_1,z_2)\mathbf{j}$ *be a bicomplex function. If the derivative*

$$f'(\omega_0) = \lim_{\substack{\omega\to\omega_0 \\ (\omega-\omega_0\ inv.)}} \frac{f(\omega)-f(\omega_0)}{\omega-\omega_0} \tag{3.2}$$

exists, then $u_x, u_y, r_x, r_y, v_x, v_y, s_x$ *and* s_y *exist, and*

$$\text{1. } f_\omega(\omega_0) = f'(\omega_0) \tag{3.3}$$

$$\text{2. } f_{\omega^{\dagger_1}}(\omega_0) = 0 \tag{3.4}$$

$$\text{3. } f_{\omega^{\dagger_2}}(\omega_0) = 0 \tag{3.5}$$

$$\text{4. } f_{\omega^{\dagger_3}}(\omega_0) = 0. \tag{3.6}$$

Moreover, if $u_x, u_y, v_x, v_y, r_x, r_y, s_x$ *and* s_y *exist, and are continuous in a neighborhood of* ω_0*, and if (3.4),(3.5) and (3.6) hold, then (3.3) exists.*

Proof. First, we remark that the complexified Cauchy-Riemann equations

$$\frac{\partial f_1}{\partial z_1} = \frac{\partial f_2}{\partial z_2}, \frac{\partial f_2}{\partial z_1} = -\frac{\partial f_1}{\partial z_2} \quad \text{at } \omega_0 \tag{3.7}$$

are equivalent to $f_{\omega^{\dagger_2}}(\omega_0) = 0$. Similarly, the following system of equations

$$\frac{\partial f_1}{\partial \bar{z}_1} = \frac{\partial f_2}{\partial \bar{z}_2}, \frac{\partial f_2}{\partial \bar{z}_1} = -\frac{\partial f_1}{\partial \bar{z}_2} \quad \text{at } \omega_0 \tag{3.8}$$

is equivalent to $f_{\omega^{\dagger_3}}(\omega_0) = 0$ and the following one

$$\frac{\partial f_1}{\partial \bar{z}_1} = -\frac{\partial f_2}{\partial \bar{z}_2}, \frac{\partial f_2}{\partial \bar{z}_1} = \frac{\partial f_1}{\partial \bar{z}_2} \quad \text{at } \omega_0 \tag{3.9}$$

is equivalent to $f_{\omega^{\dagger_1}}(\omega_0) = 0$. Now, if the bicomplex derivative $f'(\omega_0)$ exists, then from the Theorem 1 we obtain automatically that all partial derivatives exist at ω_0. Moreover, the fact that f_1 and f_2 are holomorphic in z_1 and z_2 imply that

$$\frac{\partial f_k}{\partial \bar{z}_1} = \frac{\partial f_k}{\partial \bar{z}_2} = 0 \quad \text{for } k = 1, 2. \tag{3.10}$$

Therefore $f_{\omega^{\dagger_1}}(\omega_0) = f_{\omega^{\dagger_3}}(\omega_0) = 0$ and the complexified Cauchy-Riemann equations imply $f_{\omega^{\dagger_2}}(\omega_0) = 0$ and $f_\omega(\omega_0) = f'(\omega_0)$. Conversely, if $u_x, u_y, r_x, r_y, v_x, v_y$, s_x and s_y exist, and are continuous in a neighborhood of ω_0 then $f_{\omega^{\dagger_1}}(\omega_0) = f_{\omega^{\dagger_3}}(\omega_0) = 0$ imply that f_1 and f_2 are holomorphic in z_1 and z_2. Hence, from Theorem 1, f is $\mathbb{T}$-differentiable at ω_0.□

3.2 Bicomplex Generalization of Function Theory

Our bicomplex generalization of function theory is based on the following three different representations of bicomplex numbers. The *scalar* and *vectorial* part must be adapted to each representations.

3.2.1 Class-R1

Let $a + b\mathbf{i_1} + c\mathbf{i_2} + d\mathbf{j} = z_1 + z_2\mathbf{i_2}$ where $z_1, z_2 \in \mathbb{C}(\mathbf{i_1})$. In this case, the theory will be based on assigning the part played by 1 and $\mathbf{i_2}$ to two essentially arbitrary bicomplex functions $F(\omega)$ and $G(\omega)$. We assume that these functions are defined and twice continuously differentiable in some open domain $D_0 \subset \mathbb{T}$. We require that

$$\text{Vec}\{F(\omega)^{\dagger_2}G(\omega)\} \neq 0. \tag{3.11}$$

Under this condition, (F, G) will be called a $\mathbf{i_1}$-generating pair in D_0. We remark that $\text{Vec}\{F(\omega)^{\dagger_2}G(\omega)\} = \begin{vmatrix} \text{Sc}\{F(\omega)\} & \text{Sc}\{G(\omega)\} \\ \text{Vec}\{F(\omega)\} & \text{Vec}\{G(\omega)\} \end{vmatrix}$. It follows, from Cramer's Theorem, that for every ω_0 in D_0 we can find **unique** constants $\lambda_0, \mu_0 \in \mathbb{C}(\mathbf{i_1})$ such that $w(\omega_0) = \lambda_0 F(\omega_0) + \mu_0 G(\omega_0)$. More generally we have the following result.

Theorem 4 *Let (F, G) be $\mathbf{i_1}$-generating pair in some open domain D_0. If $w(\omega) : D_0 \subset \mathbb{T} \to \mathbb{T}$, then there exist **unique** functions $\phi(\omega), \psi(\omega) : D_0 \subset \mathbb{T} \to \mathbb{C}(\mathbf{i_1})$ such that*

$$w(\omega) = \phi(\omega)F(\omega) + \psi(\omega)G(\omega) \ \forall \omega \in D_0.$$

Moreover, we have the following explicit formulas for ϕ and ψ:

$$\phi(\omega) = \frac{Vec[w(\omega)^{\dagger_2}G(\omega)]}{Vec[F(\omega)^{\dagger_2}G(\omega)]}, \ \psi(\omega) = -\frac{Vec[w(\omega)^{\dagger_2}F(\omega)]}{Vec[F(\omega)^{\dagger_2}G(\omega)]}.$$

Proof. Let (F, G) be $\mathbf{i_1}$-generating pair in some open domain D_0. Let $z_0 \in D_0$ with $w(z_0) = z_1 + z_2\mathbf{i_2}$, $F(z_0) = z_3 + z_4\mathbf{i_2}$ and $G(z_0) = z_5 + z_6\mathbf{i_2}$. In this case, $w(z_0) = \phi_2(z_0)F(z_0) + \psi_2(z_0)G(z_0)$ with $\phi_2(z_0), \psi_2(z_0) \in \mathbb{C}(\mathbf{i_1})$ if and only if $z_1 = \phi_2(z_0)z_3 + \psi_2(z_0)z_5$ and $z_2 = \phi_2(z_0)z_4 + \psi_2(z_0)z_6$. This is a well known Cramer's system of the form $AX = B$ where $A = \begin{pmatrix} z_3 & z_5 \\ z_4 & z_6 \end{pmatrix}$, $B = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ and $X = \begin{pmatrix} \phi_2(z_0) \\ \psi_2(z_0) \end{pmatrix}$. So, the unique solution is $X = A^{-1}B$ where $A^{-1} = \frac{1}{\det A}\begin{pmatrix} z_6 & -z_5 \\ -z_4 & z_3 \end{pmatrix}$. Hence, $X = \frac{1}{\text{Vec}[F(z_0)^{\dagger_2}G(z_0)]}\begin{pmatrix} z_6 & -z_5 \\ -z_4 & z_3 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \frac{1}{\text{Vec}[F(z_0)^{\dagger_2}G(z_0)]}\begin{pmatrix} z_1z_6 - z_2z_5 \\ -z_1z_4 + z_2z_3 \end{pmatrix}$
$= \frac{1}{\text{Vec}[F(z_0)^{\dagger_2}G(z_0)]}\begin{pmatrix} \text{Vec}[w(z_0)^{\dagger_2}G(z_0)] \\ -\text{Vec}[w(z_0)^{\dagger_2}F(z_0)] \end{pmatrix}$.
Then

$$\phi_2(z) = \frac{\text{Vec}[w(z)^{\dagger_2}G(z)]}{\text{Vec}[F(z)^{\dagger_2}G(z)]}, \ \psi_2(z) = -\frac{\text{Vec}[w(z)^{\dagger_2}F(z)]}{\text{Vec}[F(z)^{\dagger_2}G(z)]} \ \forall z \in D_0. \square$$

Now, we say that $w(\omega) : D_0 \subset \mathbb{T} \to \mathbb{T}$ possesses at ω_0 the $(F,G)_{\mathbf{i_1}}$-derivative $\dot{w}(\omega_0)$ if the (finite) limit

$$\dot{w}(\omega_0) = \lim_{\substack{\omega \to \omega_0 \\ (\omega - \omega_0 \ inv.)}} \frac{w(\omega) - \lambda_0 F(\omega) - \mu_0 G(\omega)}{\omega - \omega_0} \tag{3.12}$$

exists.

In the particular case where $w(\omega), F(\omega)$ and $G(\omega)$ are defined on $D_0 \subset \mathbb{C}(\mathbf{i_2}) \to \mathbb{C}(\mathbf{i_2})$ then we can find unique constants $\lambda_0, \mu_0 \in \mathbb{R}$ such that $w(\omega_0) = \lambda_0 F(\omega_0) + \mu_0 G(\omega_0)$ and we come back to the classical (in $\mathbf{i_2}$) pseudoanalytic theory developed by L. Bers and I.N. Vekua (see e.g. [1, 4, 5, 27]). In that case, using Bers's theory of Taylor series for pseudoanalytic function, V.V. Kravchenko (see [12]) obtained a locally complete system of solutions of the real stationary two-dimensional Schrödinger equation. On the other hand, in the case where $w(\omega)$, $F(\omega)$ and $G(\omega)$ are defined on $D_0 \subset \mathbb{C}(\mathbf{j}) \to \mathbb{C}(\mathbf{j})$ then we can also find unique constants $\lambda_0, \mu_0 \in \mathbb{R}$ such that $w(\omega_0) = \lambda_0 F(\omega_0) + \mu_0 G(\omega_0)$ and we are in the hyperbolic pseudoanalytic theory developed by Guo Chun Wen in [28]. Moreover, we note that if we only restrict the domain D_0 to $\mathbb{C}(\mathbf{i_2})$, the subclass of bicomplex pseudoanalytic functions obtained is precisely the class developed by V.V. Kravchenko and A. Castañeda in [6] to show that in a two-dimensional situation the Dirac equation with a scalar and an electromagnetic potentials decouples into a pair of bicomplex equations. It is also the same class of functions that used V.V. Kravchenko (see [13]) to obtain solutions of the complex stationary two-dimensional Schrödinger equation.

The following expressions are called the $\mathbf{i_1}$-characteristic coefficients of the pair (F,G) for $k = 1,2,3$:

$$a_{(F,G)}^{(k)} = -\frac{F^{\dagger_k} G_{\omega^{\dagger_k}} - F_{\omega^{\dagger_k}} G^{\dagger_k}}{F G^{\dagger_2} - F^{\dagger_2} G}, \qquad b_{(F,G)}^{(k)} = \frac{F G_{\omega^{\dagger_k}} - F_{\omega^{\dagger_k}} G}{F G^{\dagger_2} - F^{\dagger_2} G},$$

$$A_{(F,G)} = -\frac{F^{\dagger_2} G_{\omega} - F_{\omega} G^{\dagger_2}}{F G^{\dagger_2} - F^{\dagger_2} G}, \qquad B_{(F,G)} = \frac{F G_{\omega} - F_{\omega} G}{F G^{\dagger_2} - F^{\dagger_2} G}.$$

Set (for a fixed ω_0)

$$W(\omega) = w(\omega) - \lambda_0 F(\omega) - \mu_0 G(\omega), \tag{3.13}$$

the constants $\lambda_0, \mu_0 \in \mathbb{C}(\mathbf{i_1})$ being uniquely determined by the condition

$$W(\omega_0) = 0. \tag{3.14}$$

Hence, $W(\omega)$ has continuous partial derivatives if and only if $w(\omega)$ has. Moreover, $\dot{w}(\omega_0)$ exists if and only if $W'(\omega_0)$ does, and if it does exist, then $\dot{w}(\omega_0) = W'(\omega_0)$. Therefore, by Lemma 1, the existence of $W_{\omega}(\omega_0)$, $W_{\bar{\omega}}(\omega_0)$ and equations

$$W_{\omega^{\dagger_1}}(\omega_0) = 0, W_{\bar{\omega}}(\omega_0) = 0 \text{ and } W_{\omega^{\dagger_3}}(\omega_0) = 0 \tag{3.15}$$

are necessary for the existence of (3.12), and the existence and continuity of $W_\omega(\omega_0)$, $W_{\bar{\omega}}(\omega_0)$ in a neighborhood of ω_0, together with (3.15) are sufficient. Now,

$$W(\omega) = \frac{\begin{vmatrix} w(\omega) & w(\omega_0) & w(\omega_0)^{\dagger_2} \\ F(\omega) & F(\omega_0) & F(\omega_0)^{\dagger_2} \\ G(\omega) & G(\omega_0) & G(\omega_0)^{\dagger_2} \end{vmatrix}}{\begin{vmatrix} F(\omega_0) & F(\omega_0)^{\dagger_2} \\ G(\omega_0) & G(\omega_0)^{\dagger_2} \end{vmatrix}} \tag{3.16}$$

so that (3.15) may be written in the form

$$\begin{vmatrix} w_{\omega^{\dagger_k}}(\omega_0) & w(\omega_0) & w(\omega_0)^{\dagger_2} \\ F_{\omega^{\dagger_k}}(\omega_0) & F(\omega_0) & F(\omega_0)^{\dagger_2} \\ G_{\omega^{\dagger_k}}(\omega_0) & G(\omega_0) & G(\omega_0)^{\dagger_2} \end{vmatrix} = 0 \quad \text{for } k = 1, 2, 3 \tag{3.17}$$

and if (3.12) exists, then

$$\dot{w}(\omega_0) = \frac{\begin{vmatrix} w_\omega(\omega_0) & w(\omega_0) & w(\omega_0)^{\dagger_2} \\ F_\omega(\omega_0) & F(\omega_0) & F(\omega_0)^{\dagger_2} \\ G_\omega(\omega_0) & G(\omega_0) & G(\omega_0)^{\dagger_2} \end{vmatrix}}{\begin{vmatrix} F(\omega_0) & F(\omega_0)^{\dagger_2} \\ G(\omega_0) & G(\omega_0)^{\dagger_2} \end{vmatrix}}. \tag{3.18}$$

Equations (3.18) and (3.17) can be rewritten in the form

$$\dot{w} = w_\omega - A_{(F,G)} w - B_{(F,G)} w^{\dagger_2} \tag{3.19}$$

$$w_{\omega^{\dagger_k}} = a^{(k)}_{(F,G)} w + b^{(k)}_{(F,G)} w^{\dagger_2} \quad \text{for } k = 1, 2, 3. \tag{3.20}$$

Thus we have proved the following result.

Theorem 5 *Let (F,G) be a $\mathbf{i_1}$-generating pair in some open domain D_0. Every bicomplex function w defined in D_0 admits the unique representation $w = \phi F + \psi G$ where $\phi, \psi : D_0 \subset \mathbb{T} \to \mathbb{C}(\mathbf{i_1})$. Moreover, the $(F,G)_{\mathbf{i_1}}$-derivative $\dot{w} = \frac{d_{(F,G)_{\mathbf{i_1}}} w}{d\omega}$ of $w(\omega)$ exists at ω_0 and has the form*

$$\dot{w} = \phi_\omega F + \psi_\omega G = w_\omega - A_{(F,G)} w - B_{(F,G)} w^{\dagger_2} \tag{3.21}$$

if and only if

$$w_{\omega^{\dagger_1}} = a^{(1)}_{(F,G)} w + b^{(1)}_{(F,G)} w^{\dagger_2}, \tag{3.22}$$

$$w_{\omega^{\dagger_2}} = a^{(2)}_{(F,G)} w + b^{(2)}_{(F,G)} w^{\dagger_2}, \tag{3.23}$$

and

$$w_{\omega^{\dagger_3}} = a^{(3)}_{(F,G)} w + b^{(3)}_{(F,G)} w^{\dagger_2} \tag{3.24}$$

where w has continuous partial derivatives in a neighborhood of ω_0.

The equations (3.22), (3.23) and (3.24) are called the $\mathbf{i_1}$-bicomplex Vekua equations and the solutions of these equations will be the $(F,G)_{\mathbf{i_1}}$-pseudoanalytic functions.

Remark 1 *For $k = 1, 2, 3$, the equation*

$$w_{\omega^{\dagger_k}} = a^{(k)}_{(F,G)} w + b^{(k)}_{(F,G)} w^{\dagger_2} \tag{3.25}$$

can be rewritten in the following form

$$\phi_{\omega^{\dagger_k}} F + \psi_{\omega^{\dagger_k}} G = a^{(k)}_{(F,G)} w + \frac{F^{\dagger_2} G_{\omega^{\dagger_k}} - F_{\omega^{\dagger_k}} G^{\dagger_2}}{F G^{\dagger_2} - F^{\dagger_2} G} w. \tag{3.26}$$

Hence, the equation (3.25) is equivalent to

$$\phi_{\omega^{\dagger_k}} F + \psi_{\omega^{\dagger_k}} G = 0$$

if and only if

$$[G^{\dagger_k} - G^{\dagger_2}] F_{\omega^{\dagger_k}} = [F^{\dagger_k} - F^{\dagger_2}] G_{\omega^{\dagger_k}} \tag{3.27}$$

where w is not identically in the null-cone on the domain.

3.2.2 Class-R2

Let $a + b\mathbf{i_1} + c\mathbf{i_2} + d\mathbf{j} = z_1 + z_2\mathbf{i_1}$ where $z_1, z_2 \in \mathbb{C}(\mathbf{i_2})$. In this case, the theory will be based on assigning the part played by 1 and $\mathbf{i_1}$ to two essentially arbitrary bicomplex functions $F(\omega)$ and $G(\omega)$. We assume that these functions are defined and twice continuously differentiable in some open domain $D_0 \subset \mathbb{T}$. We require that

$$\text{Vec}\{F(\omega)^{\dagger_1} G(\omega)\} \neq 0. \tag{3.28}$$

Under this condition, (F,G) will be called a $\mathbf{i_2}$-generating pair in D_0. From Cramer's Theorem, it follows that for every ω_0 in D_0 we can find **unique** constants $\lambda_0, \mu_0 \in \mathbb{C}(\mathbf{i_2})$ such that $w(\omega_0) = \lambda_0 F(\omega_0) + \mu_0 G(\omega_0)$. In fact, using the same arguments than for the Theorem 4, we have the following result.

Theorem 6 *Let (F,G) be $\mathbf{i_2}$-generating pair in some open domain D_0. If $w(\omega) : D_0 \subset \mathbb{T} \to \mathbb{T}$, then there exist* ***unique*** *functions $\phi(\omega), \psi(\omega) : D_0 \subset \mathbb{T} \to \mathbb{C}(\mathbf{i_2})$ such that*

$$w(\omega) = \phi(\omega)F(\omega) + \psi(\omega)G(\omega)\ \forall \omega \in D_0.$$

Moreover, we have the following explicit formulas for ϕ and ψ:

$$\phi(\omega) = \frac{Vec[w(\omega)^{\dagger_1} G(\omega)]}{Vec[F(\omega)^{\dagger_1} G(\omega)]},\ \psi(\omega) = -\frac{Vec[w(\omega)^{\dagger_1} F(\omega)]}{Vec[F(\omega)^{\dagger_1} G(\omega)]}.$$

We say that $w(\omega) : D_0 \subset \mathbb{T} \to \mathbb{T}$ possesses at ω_0 the $(F,G)_{\mathbf{i_2}}$-derivative $\dot{w}(\omega_0)$ if the (finite) limit

$$\dot{w}(\omega_0) = \lim_{\substack{\omega \to \omega_0 \\ (\omega - \omega_0 \ inv.)}} \frac{w(\omega) - \lambda_0 F(\omega) - \mu_0 G(\omega)}{\omega - \omega_0} \tag{3.29}$$

exists. In fact, if we interchange everywhere $\mathbf{i_1}$ with $\mathbf{i_2}$, this case is exactly the same than **R1**. In particular, if we defined the function $\pi : \mathbb{T} \longrightarrow \mathbb{T}$ as

$$\pi(a + b\mathbf{i_1} + c\mathbf{i_2} + d\mathbf{j}) := a + c\mathbf{i_1} + b\mathbf{i_2} + d\mathbf{j} \tag{3.30}$$

we obtain that $w(\omega)$ possesses a $(F,G)_{\mathbf{i_1}}$-derivative at $\omega_0 \in D_0$ if and only if the function

$$(\pi \circ w \circ \pi)(\omega) \tag{3.31}$$

possesses a $(\pi \circ F \circ \pi, \pi \circ G \circ \pi)_{\mathbf{i_2}}$-derivative at $\pi(\omega_0) \in \pi(D_0)$ where

$$(\pi \circ F \circ \pi, \pi \circ G \circ \pi) \tag{3.32}$$

is a $\mathbf{i_2}$-generating pair on $\pi(D_0)$. We note that

$$\begin{aligned}
\pi(\pi(\omega)) &= \pi(\omega), && (3.33)\\
\pi(\omega_1 + \omega_2) &= \pi(\omega_1) + \pi(\omega_2), && (3.34)\\
\pi(\omega_1 \omega_2) &= \pi(\omega_1)\pi(\omega_2), && (3.35)\\
\pi\left(\frac{\omega_1}{\omega_2}\right) &= \frac{\pi(\omega_1)}{\pi(\omega_2)} \quad \text{if } \omega_2 \notin \mathcal{NC}, && (3.36)\\
\pi(w^{\dagger_3}) &= (\pi(w))^{\dagger_3}, && (3.37)\\
\pi(w^{\dagger_1}) = (\pi(w))^{\dagger_2} \quad &\text{and} \quad \pi(w^{\dagger_2}) = (\pi(w))^{\dagger_1}. && (3.38)
\end{aligned}$$

Therefore, $\left.\frac{d_{(F,G)_{\mathbf{i_2}}}(\pi \circ w \circ \pi)}{d\omega}\right|_{\omega = \pi(\omega_0)}$

$$\begin{aligned}
&= \lim_{\substack{\omega \to \pi(\omega_0) \\ (\omega - \pi(\omega_0) \ inv.)}} \frac{(\pi \circ w \circ \pi)(\omega) - \pi(\lambda_0)(\pi \circ F \circ \pi)(\omega) - \pi(\mu_0)(\pi \circ G \circ \pi)(\omega)}{\omega - \pi(\omega_0)} \\
&= \lim_{\substack{\omega \to \pi(\omega_0) \\ (\omega - \pi(\omega_0) \ inv.)}} \frac{\pi[w(\pi(\omega)) - \lambda_0 F(\pi(\omega)) - \mu_0 G(\pi(\omega))]}{\pi[\pi(\omega) - \omega_0]} \\
&= \pi\left[\lim_{\substack{\omega \to \pi(\omega_0) \\ (\omega - \pi(\omega_0) \ inv.)}} \frac{w(\pi(\omega)) - \lambda_0 F(\pi(\omega)) - \mu_0 G(\pi(\omega))}{\pi(\omega) - \omega_0}\right] \\
&= \pi\left[\left.\frac{d_{(F,G)_{\mathbf{i_1}}} w}{d\omega}\right|_{\omega = \omega_0}\right].
\end{aligned}$$

We note that the $\mathbf{i_2}$-characteristic coefficients of the pair (F,G) for $k = 1,2,3$ must be defined as:

$$a^{(k)}_{(F,G)} = -\frac{F^{\dagger_k} G_{\omega^{\dagger_k}} - F_{\omega^{\dagger_k}} G^{\dagger_k}}{FG^{\dagger_1} - F^{\dagger_1} G}, \qquad b^{(k)}_{(F,G)} = \frac{FG_{\omega^{\dagger_k}} - F_{\omega^{\dagger_k}} G}{FG^{\dagger_1} - F^{\dagger_1} G},$$

$$A_{(F,G)} = -\frac{F^{\dagger_1}G_\omega - F_\omega G^{\dagger_1}}{FG^{\dagger_1} - F^{\dagger_1}G}, \qquad B_{(F,G)} = \frac{FG_\omega - F_\omega G}{FG^{\dagger_1} - F^{\dagger_1}G}.$$

3.2.3 Class-R3

Let $a+b\mathbf{i_1}+c\mathbf{i_2}+d\mathbf{j} = z_1+z_2\mathbf{i_1}$ (resp. $z_1+z_3\mathbf{i_2}$) where $z_1, z_2, z_3 \in \mathbb{C}(\mathbf{j})$. In this case, the theory will be based on assigning the part played by 1 and $\mathbf{i_1}$ (resp. $\mathbf{i_2}$) to two essentially arbitrary bicomplex functions $F(z)$ and $G(z)$. We assume that these functions are defined and twice continuously differentiable in some open domain $D_0 \subset \mathbb{T}$. We require that

$$\mathrm{Vec}\{F(\omega)^{\dagger_3}G(\omega)\} \neq 0. \tag{3.39}$$

Under this condition, (F,G) will be called a $\mathbf{j}$-generating pair in D_0 for every ω_0 in D_0. Moreover, it will be possible to find **unique** constants $\lambda_0, \mu_0 \in \mathbb{C}(\mathbf{j})$ such that $w(\omega_0) = \lambda_0 F(\omega_0) + \mu_0 G(\omega_0)$. Here also we have the following equivalence of the Theorem 4.

Theorem 7 *Let (F,G) be $\mathbf{j}$-generating pair in some open domain D_0. If $w(\omega) : D_0 \subset \mathbb{T} \to \mathbb{T}$, then there exist* ***unique*** *functions $\phi(\omega), \psi(\omega) : D_0 \subset \mathbb{T} \to \mathbb{C}(\mathbf{j})$ such that*

$$w(\omega) = \phi(\omega)F(\omega) + \psi(\omega)G(\omega) \ \forall \omega \in D_0.$$

Moreover, we have the following explicit formulas for ϕ and ψ:

$$\phi(\omega) = \frac{Vec[w(\omega)^{\dagger_3}G(\omega)]}{Vec[F(\omega)^{\dagger_3}G(\omega)]},\ \psi(\omega) = -\frac{Vec[w(\omega)^{\dagger_3}F(\omega)]}{Vec[F(\omega)^{\dagger_3}G(\omega)]}.$$

We say that $w(\omega)$ possesses at ω_0 the $(F,G)_{\mathbf{j}}$-derivative $\dot{w}(\omega_0)$ if the (finite) limit

$$\dot{w}(\omega_0) = \lim_{\omega \to \omega_0} \frac{w(\omega) - \lambda_0 F(\omega) - \mu_0 G(\omega)}{\omega - \omega_0} \tag{3.40}$$

exists.

In this case, the following expressions are called the $\mathbf{j}$-characteristic coefficients of the pair (F,G) for $k = 1,2,3$:

$$a^{(k)}_{(F,G)} = -\frac{F^{\dagger_k}G_{\omega^{\dagger_k}} - F_{\omega^{\dagger_k}}G^{\dagger_k}}{FG^{\dagger_3} - F^{\dagger_3}G}, \qquad b^{(k)}_{(F,G)} = \frac{FG_{\omega^{\dagger_k}} - F_{\omega^{\dagger_k}}G}{FG^{\dagger_3} - F^{\dagger_3}G},$$

$$A_{(F,G)} = -\frac{F^{\dagger_3}G_\omega - F_\omega G^{\dagger_3}}{FG^{\dagger_3} - F^{\dagger_3}G}, \qquad B_{(F,G)} = \frac{FG_\omega - F_\omega G}{FG^{\dagger_3} - F^{\dagger_3}G}.$$

Now, using the same kind of arguments than for the case **R2**, we obtain the following result.

Theorem 8 *Let (F,G) be a* $\mathbf{j}$*-generating pair in some open domain* D_0*. Every bicomplex function* w *defined in* D_0 *admits the unique representation* $w = \phi F + \psi G$ *where* $\phi, \psi : D_0 \subset \mathbb{T} \to \mathbb{C}(\mathbf{j})$*. Moreover, the* $(F,G)_{\mathbf{j}}$*-derivative* $\dot{w} = \frac{d_{(F,G)_{\mathbf{j}}} w}{d\omega}$ *of* $w(\omega)$ *exists at* ω_0 *and has the form*

$$\dot{w} = \phi_\omega F + \psi_\omega G = w_\omega - A_{(F,G)} w - B_{(F,G)} w^{\dagger_3} \tag{3.41}$$

if and only if

$$w_{\omega^{\dagger_1}} = a^{(1)}_{(F,G)} w + b^{(1)}_{(F,G)} w^{\dagger_3}, \tag{3.42}$$

$$w_{\omega^{\dagger_2}} = a^{(2)}_{(F,G)} w + b^{(2)}_{(F,G)} w^{\dagger_3}, \tag{3.43}$$

and

$$w_{\omega^{\dagger_3}} = a^{(3)}_{(F,G)} w + b^{(3)}_{(F,G)} w^{\dagger_3} \tag{3.44}$$

where w *has continuous partial derivatives in a neighborhood of* ω_0*.*

The equations (3.42), (3.43) and (3.44) are called the $\mathbf{j}$-bicomplex Vekua equations and the solutions of these equations will be the $(F,G)_{\mathbf{j}}$-pseudoanalytic functions.

Remark 2 *For* $k = 1, 2, 3$*, the equation*

$$w_{\omega^{\dagger_k}} = a^{(k)}_{(F,G)} w + b^{(k)}_{(F,G)} w^{\dagger_3} \tag{3.45}$$

can be rewritten in the following form

$$\phi_{\omega^{\dagger_k}} F + \psi_{\omega^{\dagger_k}} G = a^{(k)}_{(F,G)} w + \frac{F^{\dagger_3} G_{\omega^{\dagger_k}} - F_{\omega^{\dagger_k}} G^{\dagger_3}}{F G^{\dagger_3} - F^{\dagger_3} G} w. \tag{3.46}$$

Hence, the equation (3.45) is equivalent to

$$\phi_{\omega^{\dagger_k}} F + \psi_{\omega^{\dagger_k}} G = 0$$

if and only if

$$[G^{\dagger_k} - G^{\dagger_3}] F_{\omega^{\dagger_k}} = [F^{\dagger_k} - F^{\dagger_3}] G_{\omega^{\dagger_k}} \tag{3.47}$$

where w *is not identically in the null-cone on the domain.*

In this case, it is useful to consider a more specific class of generating pair.

Definition 4 *Let* D_1 *and* D_2 *be open in* $\mathbb{C}(\mathbf{i_1})$*. Consider that* (F_{e_1}, G_{e_1}) *and* (F_{e_2}, G_{e_2}) *are complex (in* $\mathbf{i_1}$*), twice continuously differentiable, generating pairs in respectively* D_1 *and* D_2*. Under these conditions,* (F,G) *will be called a* $\mathbf{j}^*$*-generating pair in* $D_0 = D_1 \times_e D_2 \in \mathbb{T}$ *where*

$$F(z_1 + z_2\mathbf{i_2}) := F_{e_1}(z_1 - z_2\mathbf{i_1})\mathbf{e_1} + F_{e_2}(z_1 + z_2\mathbf{i_1})\mathbf{e_2} \tag{3.48}$$

and

$$G(z_1 + z_2\mathbf{i_2}) := G_{e_1}(z_1 - z_2\mathbf{i_1})\mathbf{e_1} + G_{e_2}(z_1 + z_2\mathbf{i_1})\mathbf{e_2}. \tag{3.49}$$

Lemma 2 *Let $F(\omega)$ and $G(\omega)$ two arbitrary bicomplex functions defined in some domain $D_0 \subset \mathbb{T}$. If*

$$Im\{\overline{F_{e_1}(\omega)}G_{e_1}(\omega)\} \neq 0 \ \text{or} \ Im\{\overline{F_{e_2}(\omega)}G_{e_2}(\omega)\} \neq 0 \ \forall \omega \in D_0$$

then $Vec\{F(\omega)^{\dagger_3}G(\omega)\} \neq 0 \ \forall \omega \in D_0$.

Proof. Let $F(\omega_0)^{\dagger_3}G(\omega_0) = x_1 + x_2\mathbf{i_1} + x_3\mathbf{i_2} + x_4\mathbf{j} = (x_1 + x_4\mathbf{j}) + (x_2 - x_3\mathbf{j})\mathbf{i_1} = (x_1 + x_4\mathbf{j}) + (x_3 - x_2\mathbf{j})\mathbf{i_2}$. Thus, $\text{Vec}\{F(\omega_0)^{\dagger_3}G(\omega_0)\} = 0$ if and only if $x_2 = x_3 = 0$. Moreover, $\text{Im}\{\overline{F_{e_1}(\omega_0)}G_{e_1}(z_0)\} = x_3 - x_2$ and $\text{Im}\{\overline{F_{e_2}(\omega_0)}G_{e_2}(\omega_0)\} = x_2 + x_3$. Hence, $\text{Vec}\{F(\omega_0)^{\dagger_3}G(\omega_0)\} = 0$ imply $\text{Im}\{\overline{F_{e_1}(z_0)}G_{e_1}(\omega_0)\} = 0$ and $\text{Im}\{\overline{F_{e_2}(\omega_0)}G_{e_2}(\omega_0)\} = 0 \ \forall \omega_0 \in D_0$.$\square$

Therefore, from the Lemma 2 we obtain automatically this following result.

Theorem 9 *Let $D_0 = D_1 \times_e D_2$ where D_1 and D_2 are open domains in $\mathbb{C}(\mathbf{i_1})$. If (F, G) is a $\mathbf{j}^*$-generating pair in D_0 then (F, G) is, in particular, a $\mathbf{j}$-generating pair in D_0.*

Moreover, the $\mathbf{j}^*$-generating pair will imply the following representation for a $(F, G)_\mathbf{j}$-pseudoanalytic function.

Theorem 10 *Let $D_0 = D_1 \times_e D_2$ where D_1 and D_2 are open domains in $\mathbb{C}(\mathbf{i_1})$. If (F, G) is a $\mathbf{j}^*$-generating pair in D_0 then if the function w is $(F, G)_\mathbf{j}$-pseudoanalytic on D_0 then w can be decomposed in the following way*

$$w(z_1 + z_2\mathbf{i_2}) = [w_{e_1}(z_1 - z_2\mathbf{i_1})]\mathbf{e_1} + [w_{e_2}(z_1 + z_2\mathbf{i_1})]\mathbf{e_2} \tag{3.50}$$

$\forall \ z_1 + z_2\mathbf{i_2} = (z_1 - z_2\mathbf{i_1})\mathbf{e_1} + (z_1 + z_2\mathbf{i_1})\mathbf{e_2} \in D_0$.

Proof. It is always possible to decomposed $w(z_1 + z_2\mathbf{i_2})$ in term of the idempotent representation i.e.

$$w(z_1 + z_2\mathbf{i_2}) = [w_{e_1}(z_1 + z_2\mathbf{i_2})]\mathbf{e_1} + [w_{e_2}(z_1 + z_2\mathbf{i_2})]\mathbf{e_2}.$$

Moreover, from the definition of the derivative, the function

$$W(z_1 + z_2\mathbf{i_2}) = w(z_1 + z_2\mathbf{i_2}) - \lambda_0 F(z_1 + z_2\mathbf{i_2}) - \mu_0 G(z_1 + z_2\mathbf{i_2})$$

is $\mathbb{T}$-differentiable at $z_1 + z_2\mathbf{i_2}$. Now, using the Theorem 2, we have that

$$W(z_1 + z_2\mathbf{i_2}) = [W_{e_1}(z_1 - z_2\mathbf{i_1})]\mathbf{e_1} + [W_{e_2}(z_1 + z_2\mathbf{i_1})]\mathbf{e_2}$$

and from the definition of a $\mathbf{j}^*$-generating pair we obtain that

$$\begin{aligned} w(z_1 + z_2\mathbf{i_2}) &= W_{e_1}(z_1 - z_2\mathbf{i_1})\mathbf{e_1} + W_{e_2}(z_1 + z_2\mathbf{i_1})\mathbf{e_2} \\ &+ P_1(\lambda_0)F_{e_1}(z_1 - z_2\mathbf{i_1})\mathbf{e_1} + P_2(\lambda_0)F_{e_2}(z_1 + z_2\mathbf{i_1})\mathbf{e_2} \\ &+ P_1(\mu_0)G_{e_1}(z_1 - z_2\mathbf{i_1})\mathbf{e_1} + P_2(\mu_0)G_{e_2}(z_1 + z_2\mathbf{i_1})\mathbf{e_2} \end{aligned}$$

$$
\begin{aligned}
&= [W_{e_1}(z_1 - z_2\mathbf{i_1}) + P_1(\lambda_0)F_{e_1}(z_1 - z_2\mathbf{i_1}) + P_1(\mu_0)G_{e_1}(z_1 - z_2\mathbf{i_1})]\mathbf{e_1} \\
&+ [W_{e_2}(z_1 + z_2\mathbf{i_1}) + P_2(\lambda_0)F_{e_2}(z_1 + z_2\mathbf{i_1}) + P_2(\mu_0)G_{e_2}(z_1 + z_2\mathbf{i_1})]\mathbf{e_2} \\
&= [w_{e_1}(z_1 - z_2\mathbf{i_1})]\mathbf{e_1} + [w_{e_2}(z_1 + z_2\mathbf{i_1})]\mathbf{e_2}.
\end{aligned}
$$

where $P_1(\lambda_0), P_2(\lambda_0), P_1(\mu_0), P_2(\mu_0) \in \mathbb{R}.\square$

Now, using the last theorem and the decomposition:

$$
\frac{w(z_1 + z_2\mathbf{i_2}) - \lambda_0 F(z_1 + z_2\mathbf{i_2}) - \mu_0 G(z_1 + z_2\mathbf{i_2})}{(z_1 + z_2\mathbf{i_2}) - (z_{0,1} + z_{0,2}\mathbf{i_2})}
$$

$$
\begin{aligned}
&= \frac{w_{e_1}(z_1 - z_2\mathbf{i_1}) - P_1(\lambda_0)F_{e_1}(z_1 - z_2\mathbf{i_1}) - P_1(\mu_0)G_{e_1}(z_1 - z_2\mathbf{i_1})}{(z_1 - z_2\mathbf{i_1}) - (z_{0,1} - z_{0,2}\mathbf{i_1})}\mathbf{e_1} \\
&+ \frac{w_{e_2}(z_1 + z_2\mathbf{i_1}) - P_2(\lambda_0)F_{e_2}(z_1 + z_2\mathbf{i_1}) - P_2(\mu_0)G_{e_2}(z_1 + z_2\mathbf{i_1})}{(z_1 + z_2\mathbf{i_1}) - (z_{0,1} + z_{0,2}\mathbf{i_1})}\mathbf{e_2}
\end{aligned}
$$

we obtain the following connections with the classical theory of pseudoanalytic functions.

Theorem 11 *Let $D_0 = D_1 \times_e D_2$ where D_1 and D_2 are open domains in $\mathbb{C}(\mathbf{i_1})$. If (F, G) is a $\mathbf{j}^*$-generating pair in D_0 with $w : D_0 \subset \mathbb{T} \to \mathbb{T}$ a $(F, G)_{\mathbf{j}}$-pseudoanalytic function on D_0 then*

$$
w(z_1 + z_2\mathbf{i_2}) = [w_{e_1}(z_1 - z_2\mathbf{i_1})]\mathbf{e_1} + [w_{e_2}(z_1 + z_2\mathbf{i_1})]\mathbf{e_2} \tag{3.51}
$$

where w_{e_k} is a (F_{e_k}, G_{e_k})-pseudoanalytic function on D_k for $k = 1, 2$. Moreover,

$$
\dot{w}(z_1 + z_2\mathbf{i_2}) = [\dot{w}_{e_1}(z_1 - z_2\mathbf{i_1})]\mathbf{e_1} + [\dot{w}_{e_2}(z_1 + z_2\mathbf{i_1})]\mathbf{e_2} \tag{3.52}
$$

on D_0.

Theorem 12 *If $w_{ek} : D_k \longrightarrow \mathbb{C}(\mathbf{i_1})$ is a (F_{e_k}, G_{e_k})-pseudoanalytic function on the open domain D_k for $k = 1, 2$ then the function $w : D_1 \times_e D_2 \longrightarrow \mathbb{T}$ defined as*

$$
w(z_1 + z_2\mathbf{i_2}) = w_{e1}(z_1 - z_2\mathbf{i_1})\mathbf{e_1} + w_{e2}(z_1 + z_2\mathbf{i_1})\mathbf{e_2} \ \forall\ z_1 + z_2\mathbf{i_2} \in D_1 \times_e D_2
$$

is a $(F, G)_{\mathbf{j}}$-pseudoanalytic function on $D_1 \times_e D_2$ and

$$
\dot{w}(z_1 + z_2\mathbf{i_2}) = \dot{w}_{e_1}(z_1 - z_2\mathbf{i_1})\mathbf{e_1} + \dot{w}_{e_2}(z_1 + z_2\mathbf{i_1})\mathbf{e_2}
$$

$\forall\ z_1 + z_2\mathbf{i_2} \in D_1 \times_e D_2$.

The last theorem gives another interpretation of Theorems 11 and 12 in terms of Vekua equations.

Theorem 13 *If (F_{e_1}, G_{e_1}) and (F_{e_2}, G_{e_2}) are complex (in $\mathbf{i_1}$) generating pairs in respectively D_1 and D_2. Then w is a solution on $D_0 = D_1 \times_e D_2$ of the $\mathbf{j}$-bicomplex Vekua equations with the $\mathbf{j}^*$-generating pair (F, G) if and only if $w(z_1 + z_2\mathbf{i_2}) = w_{e1}(z_1 - z_2\mathbf{i_1})\mathbf{e_1} + w_{e2}(z_1 + z_2\mathbf{i_1})\mathbf{e_2}$ where w_{e_k} is a solution on D_k of the complex (in $\mathbf{i_1}$) Vekua equation with the generating pair (F_{e_k}, G_{e_k}) for $k = 1, 2$.*

3.3 The Complexified Schrödinger Equation

Consider the equation

$$(\triangle_{\mathbb{C}} - \nu(z_1, z_2))f = 0 \tag{3.53}$$

in $\Omega \subset \mathbb{R}^4$, where $\triangle_{\mathbb{C}} = \partial^2_{z_1} + \partial^2_{z_2}$, ν and f are complex (in $\mathbf{i_1}$) valued functions. The equation (3.53) is simply the complexification of the two-dimensional stationary Schrödinger equation where $\triangle_{\mathbb{C}}$ is the complex Laplacian (see [8, 9, 10, 16]).

3.3.1 The Complex Laplacian

First of all, we will write the complex Laplacian in a more explicit way to see that it contains in the same time the classical Laplacian operator and the wave operator.

Lemma 3 *Let $\omega = z_1 + z_2\mathbf{i_2}$, where $z_1, z_2 \in \mathbb{C}(\mathbf{i_1})$ then*

$$\partial_\omega \partial_{\bar{\omega}} = \frac{1}{4}(\partial^2_{z_1} + \partial^2_{z_2}) = \frac{1}{4}\triangle_{\mathbb{C}}$$

$\forall f \in C^2(\Omega)$ where Ω is an open set in $\mathbb{R}^4$.

Proof. Let $\partial_\omega = \frac{1}{2}\left(\partial_{z_1} - \mathbf{i_2}\partial_{z_2}\right)$ and $\partial_{\bar{\omega}} = \frac{1}{2}\left(\partial_{z_1} + \mathbf{i_2}\partial_{z_2}\right)$ then

$$\begin{aligned} 4\partial_\omega \partial_{\bar{\omega}} &= \partial_{z_1}\left(\partial_{z_1} + \mathbf{i_2}\partial_{z_2}\right) - \mathbf{i_2}\partial_{z_2}\left(\partial_{z_1} + \mathbf{i_2}\partial_{z_2}\right) \\ &= \partial^2_{z_1} + \mathbf{i_2}\partial^2_{z_1 z_2} - \mathbf{i_2}\partial^2_{z_2 z_1} + \partial^2_{z_2} \\ &= \partial^2_{z_1} + \partial^2_{z_2}. \square \end{aligned}$$

Proposition 1 *Let $\partial_{z_1} = \frac{1}{2}\left(\partial_x - \mathbf{i_1}\partial_y\right)$ and $\partial_{z_2} = \frac{1}{2}\left(\partial_p - \mathbf{i_1}\partial_q\right)$ then*

$$16\partial_\omega \partial_{\bar{\omega}} = 4\triangle_{\mathbb{C}} = \left(\partial^2_x - \partial^2_y + \partial^2_p - \partial^2_q\right) - 2\mathbf{i_1}\left(\partial^2_{xy} + \partial^2_{pq}\right) \tag{3.54}$$

$\forall f \in C^2(\Omega)$ where Ω is an open set in $\mathbb{R}^4$.

Proof. Consider,

$$\begin{aligned} 4\partial^2_{z_1} &= \partial_x\left(\partial_x - \mathbf{i_1}\partial_y\right) - \mathbf{i_1}\partial_y\left(\partial_x - \mathbf{i_1}\partial_y\right) \\ &= \partial^2_x - \mathbf{i_1}\partial^2_{xy} - \mathbf{i_1}\partial^2_{yx} - \partial^2_y \\ &= \partial^2_x - \partial^2_y - 2\mathbf{i_1}\partial^2_{xy}. \end{aligned}$$

Therefore,

$$4\left(\partial^2_{z_1} + \partial^2_{z_2}\right) = \left(\partial^2_x - \partial^2_y + \partial^2_p - \partial^2_q\right) - 2\mathbf{i_1}\left(\partial^2_{xy} + \partial^2_{pq}\right). \square$$

Remark 3 *In the last proposition, if we let y and q be constant variables, then*

1. *$\Omega \subset \mathbb{C}(\mathbf{i_2})$;*

2. $4\partial_\omega\partial_{\bar\omega} = \partial_z\partial_{\bar z}$ *where* $\partial_{\bar z} = \frac{1}{2}\left(\partial_x + \mathbf{i_2}\partial_p\right)$ *and* $\partial_z = \frac{1}{2}\left(\partial_x - \mathbf{i_2}\partial_p\right)$*;*

3. $4\triangle_{\mathbb{C}} = 4\partial_z\partial_{\bar z} = \partial_x^2 + \partial_p^2 = \triangle$, ***the Laplacian operator****.*

Similarly, if y and p are constant variables, then

1. $\Omega \subset \mathbb{D}$*;*

2. $4\partial_\omega\partial_{\bar\omega} = \partial_z\partial_{\bar z}$ *where* $\partial_{\bar z} = \frac{1}{2}\left(\partial_x - \mathbf{j}\partial_q\right)$ *and* $\partial_z = \frac{1}{2}\left(\partial_x + \mathbf{j}\partial_q\right)$*;*

3. $4\triangle_{\mathbb{C}} = 4\partial_z\partial_{\bar z} = \partial_x^2 - \partial_q^2 = \Box$, ***the wave operator****.*

3.3.2 Factorization of the Complexified Schrödinger Operator

It is well known that if f_0 is a nonvanishing particular solution of the one-dimensional stationary Schrödinger equation

$$\left(-\frac{d^2}{dx^2} + \nu(x)\right)$$

then the Scrödinger operator can be factorized as follows:

$$-\frac{d^2}{dx^2} + \nu(x) = \left(\frac{d}{dx} + \frac{f_0'}{f_0}\right)\left(\frac{d}{dx} - \frac{f_0'}{f_0}\right).$$

The next result gives the analogue for the complexified Schrödinger operator. By C we denote the $\dagger_2$-bicomplex conjugation operator.

Theorem 14 *Let $f_0 : \Omega \subset \mathbb{R}^4 \longrightarrow \mathbb{C}(\mathbf{i_1})$ be a nonvanishing particular solution of (3.53). Then for any $\mathbb{C}(\mathbf{i_1})$-valued continuously twice differentiable function φ the following equality hold:*

$$(\triangle_{\mathbb{C}} - \nu)\varphi = 4\left(\partial_{\bar\omega} + \frac{\partial_\omega f_0}{f_0}C\right)\left(\partial_\omega - \frac{\partial_\omega f_0}{f_0}C\right)\varphi. \tag{3.55}$$

Proof. Let $f_0 : \Omega \subset \mathbb{R}^4 \longrightarrow \mathbb{C}(\mathbf{i_1})$ be a nonvanishing particular solution of (3.53). Then

$$\begin{aligned}
\left(\partial_{\bar\omega} + \frac{\partial_\omega f_0}{f_0}C\right)\left(\partial_\omega - \frac{\partial_\omega f_0}{f_0}C\right)\varphi &= \left(\partial_{\bar\omega} + \frac{\partial_\omega f_0}{f_0}C\right)\left(\partial_\omega\varphi - \frac{\partial_\omega f_0}{f_0}\varphi\right) \\
&= \partial_{\bar\omega}\partial_\omega\varphi - \partial_{\bar\omega}\left(\frac{\partial_\omega f_0}{f_0}\varphi\right) + \frac{\partial_\omega f_0}{f_0}\partial_{\bar\omega}\varphi - \frac{|\partial_\omega f_0|^2_{\mathbf{i_1}}}{f_0^2}\varphi \\
&= \frac{1}{4}\triangle_{\mathbb{C}}\varphi - \partial_{\bar\omega}\left(\frac{\partial_\omega f_0}{f_0}\right)\varphi - \frac{|\partial_\omega f_0|^2_{\mathbf{i_1}}}{f_0^2}\varphi \\
&= \frac{1}{4}\triangle_{\mathbb{C}}\varphi - \left(\frac{\partial_{\bar\omega}\partial_\omega f_0 \cdot f_0 - |\partial_\omega f_0|^2_{\mathbf{i_1}}}{f_0^2}\right)\varphi - \frac{|\partial_\omega f_0|^2_{\mathbf{i_1}}}{f_0^2}\varphi \\
&= \frac{1}{4}\triangle_{\mathbb{C}}\varphi - \frac{\frac{1}{4}\triangle_{\mathbb{C}} f_0}{f_0^2}\varphi.
\end{aligned}$$

However, $(\triangle_{\mathbb{C}} - \nu) f_0 = 0 \Rightarrow \frac{\triangle_{\mathbb{C}} f_0}{f_0} = \nu$. Hence,

$$(\triangle_{\mathbb{C}} - \nu)\varphi = 4\left(\partial_{\bar{\omega}} + \frac{\partial_\omega f_0}{f_0} C\right)\left(\partial_\omega - \frac{\partial_\omega f_0}{f_0} C\right)\varphi.\Box \tag{3.56}$$

Remark 4 *From the Remark 3, we see that the complexified Schrödinger equation contains the stationary two-dimensional Schrödinger equation*

$$(\triangle - \nu(x,p))f = 0$$

and the Klein-Gordon equation

$$(\Box - \nu(x,q))f = 0.$$

Hence, our factorization of the complexified Schrödinger equation is a generalization of the factorization obtained in [12] for the stationary two-dimensional Schrödinger equation and for the factorization obtained in [15] for the Klein-Gordon equation.

3.3.3 Relationship Between Bicomplex Generalized Analytic Functions and Solutions of the Complexified Schrödinger Equation

The next Lemma has been inspired from a similar result in the complex plane (see [2], p 140).

Lemma 4 *Let $b : \Omega \subset \mathbb{R}^4 \longrightarrow \mathbb{T}$ be a bicomplex function such that b_ω is $\mathbb{C}(\mathbf{i_1})$-valued, and let $W = u + \mathbf{i_2} v : \Omega \subset \mathbb{R}^4 \longrightarrow \mathbb{T}$ be a solution of the equation*

$$W_{\omega^{\dagger_2}} = bW^{\dagger_2} \text{ on } \Omega. \tag{3.57}$$

Thus, $u : \Omega \subset \mathbb{R}^4 \longrightarrow \mathbb{C}(\mathbf{i_1})$ is a solution of the equation

$$\partial_{\omega^{\dagger_2}} \partial_\omega u - (|b|^2_{\mathbf{i_1}} + b_\omega)u = 0 \text{ on } \Omega \tag{3.58}$$

and $v : \Omega \subset \mathbb{R}^4 \longrightarrow \mathbb{C}(\mathbf{i_1})$ is a solution of the equation

$$\partial_{\omega^{\dagger_2}} \partial_\omega v - (|b|^2_{\mathbf{i_1}} - b_\omega)v = 0 \text{ on } \Omega. \tag{3.59}$$

Proof. Using the $\dagger_2$ on both sides of Eq. (3.57), we have that

$$\partial_{\omega^{\dagger_2}}(u + \mathbf{i_2} v) = b(u - \mathbf{i_2} v) \Leftrightarrow \partial_\omega (u - \mathbf{i_2} v) = b^{\dagger_2}(u + \mathbf{i_2} v) \text{ on } \Omega. \tag{3.60}$$

Therefore,

$$\begin{aligned}
\partial_\omega \partial_{\omega^{\dagger_2}}(u + \mathbf{i_2} v) &= \partial_\omega b \cdot (u - \mathbf{i_2} v) + b\partial_\omega(u - \mathbf{i_2} v) \\
&= b_\omega (u - \mathbf{i_2} v) + bb^{\dagger_2}(u + \mathbf{i_2} v) \\
&= b_\omega (u - \mathbf{i_2} v) + |b|^2_{\mathbf{i_1}}(u + \mathbf{i_2} v) \text{ on } \Omega.
\end{aligned}$$

Now, by equality of the scalar and vectorial part, we obtain the Equations (3.58) and (3.59).$\Box$

Theorem 15 *Let W be a solution of the following bicomplex Vekua equation*

$$\left(\partial_{\omega^{\dagger_2}} - \frac{\partial_{\omega^{\dagger_2}} f_0}{f_0} C\right) W = 0 \tag{3.61}$$

where f_0 is a nonvanishing solution of the complexified Schrödinger equation (3.53). Then $u = Sc(W)$ is a solution of (3.53) and $v = Vec(W)$ is a solution of the equation

$$\left(\triangle_{\mathbb{C}} + \nu(z_1, z_2) - 2\left(\frac{|\bigtriangledown_{\mathbb{C}} f_0|_{\mathbf{i_1}}}{f_0}\right)^2\right) v = 0 \tag{3.62}$$

where $\bigtriangledown_{\mathbb{C}} = \partial_{z_1} + \mathbf{i_2}\partial_{z_2}$.

Proof. Consider the function $b = \frac{\partial_{\omega^{\dagger_2}} f_0}{f_0}$. Then

$$\begin{aligned} b_\omega = \partial_\omega \left(\frac{\partial_{\omega^{\dagger_2}} f_0}{f_0}\right) &= \frac{(\partial_\omega \partial_{\omega^{\dagger_2}} f_0) \cdot f_0 - (\partial_{\omega^{\dagger_2}} f_0)(\partial_\omega f_0)}{f_2^2} \\ &= \frac{\triangle_{\mathbb{C}} f_0}{4 f_0} - \frac{|\partial_{\omega^{\dagger_2}} f_0|^2_{\mathbf{i_1}}}{f_2^2} \\ &= \frac{\nu(z_1, z_2)}{4} - \frac{|\partial_{\omega^{\dagger_2}} f_0|^2_{\mathbf{i_1}}}{f_2^2}. \end{aligned}$$

Therefore, b_ω is a $\mathbb{C}(\mathbf{i_1})$-valued function. Now, from Lemma 4,

$$\frac{\triangle_{\mathbb{C}} u}{4} = \left(\frac{\nu(z_1, z_2)}{4} - \frac{|\partial_{\omega^{\dagger_2}} f_0|^2_{\mathbf{i_1}}}{f_2^2} + \frac{|\partial_{\omega^{\dagger_2}} f_0|^2_{\mathbf{i_1}}}{f_2^2}\right) u$$

i.e

$$(\triangle_{\mathbb{C}} - \nu) u = 0,$$

and

$$\begin{aligned} \frac{\triangle_{\mathbb{C}} v}{4} &= \left(-\frac{\nu(z_1, z_2)}{4} + \frac{|\partial_{\omega^{\dagger_2}} f_0|^2_{\mathbf{i_1}}}{f_2^2} + \frac{|\partial_{\omega^{\dagger_2}} f_0|^2_{\mathbf{i_1}}}{f_2^2}\right) v \\ &= \left(-\frac{\nu(z_1, z_2)}{4} + \frac{2|\partial_{\omega^{\dagger_2}} f_0|^2_{\mathbf{i_1}}}{f_2^2}\right) v \\ &= \left(-\frac{\nu(z_1, z_2)}{4} + \frac{(\partial_{z_1} f_0)^2 + (\partial_{z_2} f_0)^2}{2 f_2^2}\right) v \end{aligned}$$

i.e.

$$(\triangle_{\mathbb{C}} - \eta)\, v = 0$$

where $\eta(z_1, z_2) = -\nu(z_1, z_2) + \frac{2|\bigtriangledown_{\mathbb{C}} f_0|^2_{\mathbf{i_1}}}{f_2^2}$. $\square$

Remark 5 *From Theorem 5, if W possesses a $(f_0, \frac{\mathbf{i_2}}{f_0})_{\mathbf{i_1}}$-derivative on an open set $\Omega \subset \mathbb{T}$ then W is a solution of the bicomplex Vekua equation (3.61):*

$$\left(\partial_{\omega^{\dagger_2}} - \frac{\partial_{\omega^{\dagger_2}} f_0}{f_0} C\right) W = 0 \ \text{on}\ \Omega.$$

In that case, $a^{(2)}_{(F,G)} = 0$ and $b^{(2)}_{(F,G)} = \frac{\partial_{\omega^{\dagger_2}} f_0}{f_0}$ where

$$F = f_0 \quad \text{and} \quad G = \frac{\mathbf{i_2}}{f_0}$$

is a $\mathbf{i_1}$-generating pair for (3.61).

From the last remark, we can conclude that the bicomplex pseudoanalytic function theory open the way to find explicit solutions of the complexified Schrödinger equation.

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