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On Factorization of Bicomplex Meromorphic Functions

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Abstract. In this paper the factorization theory of meromorphic functions of one complex variable is promoted to bicomplex meromorphic functions. Many results of one complex variable case are seen to hold in bicomplex case, and it is found that there are results for meromorphic functions of one complex variable which are not true for bicomplex meromorphic functions. In particular, we show that for any bicomplex transcendental meromorphic function F, there exists a bicomplex meromorphic function G such that GF is prime even if the set:

$$\{a \in \mathbb{T} : F(w) + a\phi(w) \text{ is not prime}\}$$

is empty or of cardinality $\aleph_1$ for any non-constant fractional linear bicomplex function ϕ. Moreover, as specific application, we obtain six additional possible forms of factorization of the complex cosine $\cos z$ in the bicomplex space.

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1. Introduction

The factorization theory of meromorphic functions of one complex variable is to study how a given meromorphic function can be factorized into other simpler meromorphic functions in the sense of composition. In number theory, every natural number can be factorized as a product of prime numbers. Therefore, prime numbers serve as building blocks of natural numbers and the theory of prime numbers is one of the main subarea of number theory. In our situation, we also have the so-called prime functions which play a similar role in the factorization theory of meromorphic functions as prime numbers do in number theory. More specifically, factorization theory of meromorphic functions essentially deals with the primeness,

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pseudo-primeness and unique factorizability of a meromorphic function. We start with the following concepts.

Definition 1.1. Let F be a meromorphic function. Then an expression

$$F(z) = f(g(z)) \tag{1.1}$$

where f is meromorphic and g is entire (g may be meromorphic when f is a rational function) is called a factorization of F with f and g as its left and right factors respectively. F is said to be non-factorizable or prime if for every representation of F of the form (1.1) we have that either f or g is linear. If every representation of F of the form (1.1) implies that f is rational or g is a polynomial (f is linear whenever g is transcendental, g is linear whenever f is transcendental), we say that F is pseudo-prime (left-prime, right-prime). If the factors are restricted to entire functions, the factorization is said to be in entire sense and we have the corresponding concepts of primeness in entire sense (called E-primeness), pseudo-primeness in entire sense (called E-pseudo-primeness) etc.

The first example of prime function is $F(z) = exp(z)+z$ given by Rosenbloom [23] who gave the definition of prime transcendental entire function by considering entire factors only, and asserted without proof that the function $F(z) = z+exp(z)$ is prime. In 1968, F. Gross [7] gave a complete proof of this assertion and extended the study of primeness to meromorphic functions and gave Definition 1.1. No systematic theory has actually been developed to handle the problems of factorization of transcendental meromorphic functions. However, recently, T.W. Ng [9, 10, 11, 12] proved some results which of course can solve some factorization problems in a systematic way. He introduced the methods from the Theory of Complex Analytic Sets and local holomorphic dynamics to solve some factorization problems. Classical function theory and the Nevanlinna Value Distribution theory are the main tools used in factorization theory of meromorphic functions. Most of the classes of functions which have been studied are concerned with the following one for several factors: (1) Growth of the function, (2) Distribution of zeros, (3) Periodicity, (4) Fixed-points, (5) Solutions of linear differential equations. For complete details on factorization theory of meromorphic functions one can refer to the books of C.T. Chuang and C.C. Yang [3], and F. Gross [5].

The main purpose of the present paper is to try to extend and promote the research on Factorization theory of meromorphic functions in one complex variable to two complex variables via Bicomplex Function Theory [14, 15, 16, 18]. In our study of factorization theory of bicomplex meromorphic functions, the idempotent representation of bicomplex meromorphic functions plays a vital role since the parallel definitions of factorization of meromorphic functions of one complex variable do not work. It is found that many results from one variable theory hold in bicomplex situation whereas some fail to hold, and this is the point of difference between the two situations and so makes sense to investigate. In particular, we show that for any bicomplex transcendental meromorphic function F, there exists a bicomplex meromorphic function G such that GF is prime even if the set:

$$\{a \in \mathbb{T} : F(w) + a\phi(w) \text{ is not prime}\}$$

is empty or of cardinality $\aleph_1$ for any non-constant fractional linear bicomplex function ϕ. Moreover, as specific application, we obtain six additional possible forms of factorization of the complex cosine $\cos z$ in the bicomplex space.

2. Preliminaries

2.1. Bicomplex numbers

Bicomplex numbers are defined as

$$\mathbb{T} := \{z_1 + z_2\mathbf{i_2} \mid z_1, z_2 \in \mathbb{C}(\mathbf{i_1})\} \tag{2.1}$$

where the imaginary units $\mathbf{i_1}, \mathbf{i_2}$ and $\mathbf{j}$ are governed by the rules: $\mathbf{i_1}^2 = \mathbf{i_2}^2 = -1$, $\mathbf{j}^2 = 1$ and

$$\begin{array}{rclcl} \mathbf{i_1}\mathbf{i_2} & = & \mathbf{i_2}\mathbf{i_1} & = & \mathbf{j}, \\ \mathbf{i_1}\mathbf{j} & = & \mathbf{j}\mathbf{i_1} & = & -\mathbf{i_2}, \\ \mathbf{i_2}\mathbf{j} & = & \mathbf{j}\mathbf{i_2} & = & -\mathbf{i_1}. \end{array} \tag{2.2}$$

Note that we define $\mathbb{C}(\mathbf{i}_k) := \{x + y\mathbf{i}_k \mid \mathbf{i}_k^2 = -1 \text{ and } x, y \in \mathbb{R}\}$ for $k = 1, 2$. Hence, it is easy to see that the multiplication of two bicomplex numbers is commutative. In fact, the bicomplex numbers

$$\mathbb{T} \cong \mathrm{Cl}_{\mathbb{C}}(1,0) \cong \mathrm{Cl}_{\mathbb{C}}(0,1)$$

are *unique* among the complex Clifford algebras in that they are commutative but not division algebra. It is also convenient to write the set of bicomplex numbers as

$$\mathbb{T} := \{w_0 + w_1\mathbf{i_1} + w_2\mathbf{i_2} + w_3\mathbf{j} \mid w_0, w_1, w_2, w_3 \in \mathbb{R}\}. \tag{2.3}$$

In particular, in equation (2.1), if we put $z_1 = x$ and $z_2 = y\mathbf{i_1}$ with $x, y \in \mathbb{R}$, then we obtain the following subalgebra of hyperbolic numbers, also called duplex numbers (see, e.g., [20, 26]):

$$\mathbb{D} := \{x + y\mathbf{j} \mid \mathbf{j}^2 = 1,\ x, y \in \mathbb{R}\} \cong \mathrm{Cl}_{\mathbb{R}}(0,1).$$

Complex conjugation plays an important role both for algebraic and geometric properties of $\mathbb{C}$. For bicomplex numbers, there are three possible conjugations. Let $w \in \mathbb{T}$ and $z_1, z_2 \in \mathbb{C}(\mathbf{i_1})$ such that $w = z_1 + z_2\mathbf{i_2}$. Then we define the three conjugations as:

$$w^{\dagger_1} = (z_1 + z_2\mathbf{i_2})^{\dagger_1} := \overline{z}_1 + \overline{z}_2\mathbf{i_2}, \tag{2.4a}$$

$$w^{\dagger_2} = (z_1 + z_2\mathbf{i_2})^{\dagger_2} := z_1 - z_2\mathbf{i_2}, \tag{2.4b}$$

$$w^{\dagger_3} = (z_1 + z_2\mathbf{i_2})^{\dagger_3} := \overline{z}_1 - \overline{z}_2\mathbf{i_2}, \tag{2.4c}$$

where $\overline{z}_k$ is the standard complex conjugate of complex numbers $z_k \in \mathbb{C}(\mathbf{i_1})$. If we say that the bicomplex number $w = z_1 + z_2\mathbf{i_2} = w_0 + w_1\mathbf{i_1} + w_2\mathbf{i_2} + w_3\mathbf{j}$ has the "signature" $(++++)$, then the conjugations of type 1,2 or 3 of w have, respectively, the signatures $(+-+-)$, $(++--)$ and $(+--+)$. We can verify easily that the composition of the conjugates gives the four-dimensional abelian Klein group:

$$\begin{array}{|c||c|c|c|c|}\hline \circ & \dagger_0 & \dagger_1 & \dagger_2 & \dagger_3 \\ \hline\hline \dagger_0 & \dagger_0 & \dagger_1 & \dagger_2 & \dagger_3 \\ \hline \dagger_1 & \dagger_1 & \dagger_0 & \dagger_3 & \dagger_2 \\ \hline \dagger_2 & \dagger_2 & \dagger_3 & \dagger_0 & \dagger_1 \\ \hline \dagger_3 & \dagger_3 & \dagger_2 & \dagger_1 & \dagger_0 \\ \hline \end{array} \tag{2.5}$$

where $w^{\dagger_0} := w \ \forall w \in \mathbb{T}$.

The three kinds of conjugation all have some of the standard properties of conjugations, such as:

$$(s+t)^{\dagger_k} = s^{\dagger_k} + t^{\dagger_k}, \tag{2.6}$$

$$\left(s^{\dagger_k}\right)^{\dagger_k} = s, \tag{2.7}$$

$$(s \cdot t)^{\dagger_k} = s^{\dagger_k} \cdot t^{\dagger_k}, \tag{2.8}$$

for $s, t \in \mathbb{T}$ and $k = 0, 1, 2, 3$.

We know that the product of a standard complex number with its conjugate gives the square of the Euclidean metric in $\mathbb{R}^2$. The analogs of this, for bicomplex numbers, are the following. Let $z_1, z_2 \in \mathbb{C}(\mathbf{i_1})$ and $w = z_1 + z_2\mathbf{i_2} \in \mathbb{T}$, then we have that [20]:

$$|w|^2_{\mathbf{i_1}} := w \cdot w^{\dagger_2} = z_1^2 + z_2^2 \in \mathbb{C}(\mathbf{i_1}), \tag{2.9a}$$

$$|w|^2_{\mathbf{i_2}} := w \cdot w^{\dagger_1} = \left(|z_1|^2 - |z_2|^2\right) + 2\mathrm{Re}(z_1\overline{z}_2)\mathbf{i_2} \in \mathbb{C}(\mathbf{i_2}), \tag{2.9b}$$

$$|w|^2_{\mathbf{j}} := w \cdot w^{\dagger_3} = \left(|z_1|^2 + |z_2|^2\right) - 2\mathrm{Im}(z_1\overline{z}_2)\mathbf{j} \in \mathbb{D}, \tag{2.9c}$$

where the subscript of the square modulus refers to the subalgebra $\mathbb{C}(\mathbf{i_1}), \mathbb{C}(\mathbf{i_2})$ or $\mathbb{D}$ of $\mathbb{T}$ in which w is projected. Note that for $z_1, z_2 \in \mathbb{C}(\mathbf{i_1})$ and $w = z_1 + z_2\mathbf{i_2} \in \mathbb{T}$, we can define the usual (Euclidean in $\mathbb{R}^4$) norm of w as $|w| = \sqrt{|z_1|^2 + |z_2|^2} = \sqrt{\mathrm{Re}(|w|^2_{\mathbf{j}})}$.

It is easy to verify that $w \cdot \dfrac{w^{\dagger_2}}{|w|^2_{\mathbf{i_1}}} = 1$. Hence, the inverse of w is given by

$$w^{-1} = \frac{w^{\dagger_2}}{|w|^2_{\mathbf{i_1}}}. \tag{2.10}$$

From this, we find that the set $\mathcal{NC}$ of zero divisors of $\mathbb{T}$, called the *null-cone*, is given by $\{z_1 + z_2\mathbf{i_2} \mid z_1^2 + z_2^2 = 0\}$, which can be rewritten as

$$\mathcal{NC} = \{z(\mathbf{i_1} \pm \mathbf{i_2}) |\ z \in \mathbb{C}(\mathbf{i_1})\}. \tag{2.11}$$

2.2. Bicomplex holomorphic functions

It is also possible to define differentiability of a function at a point of $\mathbb{T}$:

Definition 2.1. Let U be an open set of $\mathbb{T}$ and $w_0 \in U$. Then, $f : U \subseteq \mathbb{T} \longrightarrow \mathbb{T}$ is said to be $\mathbb{T}$-differentiable at w_0 with derivative equal to $f'(w_0) \in \mathbb{T}$ if

$$\lim_{\substack{w \to w_0 \\ (w-w_0 \ inv.)}} \frac{f(w) - f(w_0)}{w - w_0} = f'(w_0).$$

We also say that the function f is $\mathbb{T}$-holomorphic on an open set U if and only if f is $\mathbb{T}$-differentiable at each point of U.

Using $w = z_1 + z_2\mathbf{i_2}$, a bicomplex number w can be seen as an element (z_1, z_2) of $\mathbb{C}^2$, so a function $f(z_1 + z_2\mathbf{i_2}) = f_1(z_1, z_2) + f_2(z_1, z_2)\mathbf{i_2}$ of $\mathbb{T}$ can be seen as a mapping $f(z_1, z_2) = (f_1(z_1, z_2), f_2(z_1, z_2))$ of $\mathbb{C}^2$. Here we have a characterization of such mappings:

Theorem 2.2. *Let U be an open set and $f : U \subseteq \mathbb{T} \longrightarrow \mathbb{T}$ such that $f \in C^1(U)$, and let $f(z_1 + z_2\mathbf{i_2}) = f_1(z_1, z_2) + f_2(z_1, z_2)\mathbf{i_2}$. Then f is $\mathbb{T}$-holomorphic on U if and only if*

$$f_1 \text{ and } f_2 \text{ are holomorphic in } z_1 \text{ and } z_2,$$

and

$$\frac{\partial f_1}{\partial z_1} = \frac{\partial f_2}{\partial z_2} \text{ and } \frac{\partial f_2}{\partial z_1} = -\frac{\partial f_1}{\partial z_2} \text{ on } U.$$

Moreover, $f' = \frac{\partial f_1}{\partial z_1} + \frac{\partial f_2}{\partial z_1}\mathbf{i_2}$ and $f'(w)$ is invertible if and only if $\det \mathcal{J}_f(w) \neq 0$.

This theorem can be obtained from results in [14] and [19]. Moreover, by the Hartogs theorem [25], it is possible to show that "$f \in C^1(U)$" can be dropped from the hypotheses. Hence, it is natural to define the corresponding class of mappings for $\mathbb{C}^2$:

Definition 2.3. The class of $\mathbb{T}$-holomorphic mappings on a open set $U \subseteq \mathbb{C}^2$ is defined as follows:

$$TH(U) := \{f{:}U \subseteq \mathbb{C}^2 \longrightarrow \mathbb{C}^2 | f \in \mathrm{H}(U) \text{ and } \frac{\partial f_1}{\partial z_1} = \frac{\partial f_2}{\partial z_2}, \frac{\partial f_2}{\partial z_1} = -\frac{\partial f_1}{\partial z_2} \text{ on } U\}.$$

It is the subclass of holomorphic mappings of $\mathbb{C}^2$ satisfying the complexified Cauchy-Riemann equations.

We remark that $f \in TH(U)$ in terms of $\mathbb{C}^2$ if and only if f is $\mathbb{T}$-differentiable on U. It is also important to know that every bicomplex number $z_1 + z_2\mathbf{i_2}$ has the following unique idempotent representation:

$$z_1 + z_2\mathbf{i_2} = (z_1 - z_2\mathbf{i_1})\mathbf{e_1} + (z_1 + z_2\mathbf{i_1})\mathbf{e_2}. \tag{2.12}$$

where $\mathbf{e_1} = \frac{1+\mathbf{j}}{2}$ and $\mathbf{e_2} = \frac{1-\mathbf{j}}{2}$. This representation is very useful because: addition, multiplication and division can be done term-by-term. It is also easy to verify the following characterization of the non-invertible elements.

Proposition 2.4. *An element $w = z_1 + z_2\mathbf{i_2}$ will be in the null-cone if and only if $z_1 - z_2\mathbf{i_1} = 0$ or $z_1 + z_2\mathbf{i_1} = 0$.*

The notion of holomorphicity can also be seen with this kind of notation. For this we need to define the projections $P_1, P_2 : \mathbb{T} \longrightarrow \mathbb{C}(\mathbf{i_1})$ as $P_1(z_1 + z_2\mathbf{i_2}) = z_1 - z_2\mathbf{i_1}$ and $P_2(z_1 + z_2\mathbf{i_2}) = z_1 + z_2\mathbf{i_1}$.

Definition 2.5. We say that $X \subseteq \mathbb{T}$ is a $\mathbb{T}$-cartesian set determined by X_1 and X_2 if $X = X_1 \times_e X_2 := \{z_1 + z_2\mathbf{i_2} \in \mathbb{T} : z_1 + z_2\mathbf{i_2} = w_1\mathbf{e_1} + w_2\mathbf{e_2}, (w_1, w_2) \in X_1 \times X_2\}$.

Now, it is possible to state the following striking theorems [14]:

Theorem 2.6. *Let X_1 and X_2 be open sets in $\mathbb{C}(\mathbf{i_1})$. If $f_{e1} : X_1 \longrightarrow \mathbb{C}(\mathbf{i_1})$ and $f_{e2} : X_2 \longrightarrow \mathbb{C}(\mathbf{i_1})$ are holomorphic functions of $\mathbb{C}(\mathbf{i_1})$ on X_1 and X_2 respectively, then the function $f : X_1 \times_e X_2 \longrightarrow \mathbb{T}$ defined as*

$$f(z_1 + z_2\mathbf{i_2}) = f_{e1}(z_1 - z_2\mathbf{i_1})\mathbf{e_1} + f_{e2}(z_1 + z_2\mathbf{i_1})\mathbf{e_2} \ \forall \ z_1 + z_2\mathbf{i_2} \in X_1 \times_e X_2$$

is $\mathbb{T}$-holomorphic on the open set $X_1 \times_e X_2$ and

$$f'(z_1 + z_2\mathbf{i_2}) = f'_{e1}(z_1 - z_2\mathbf{i_1})\mathbf{e_1} + f'_{e2}(z_1 + z_2\mathbf{i_1})\mathbf{e_2}$$

$\forall \ z_1 + z_2\mathbf{i_2} \in X_1 \times_e X_2$.

Theorem 2.7. *Let X be an open set in $\mathbb{T}$, and let $f : X \longrightarrow \mathbb{T}$ be a $\mathbb{T}$-holomorphic function on X. Then there exist holomorphic functions $f_{e1} : X_1 \longrightarrow \mathbb{C}(\mathbf{i_1})$ and $f_{e2} : X_2 \longrightarrow \mathbb{C}(\mathbf{i_1})$ with $X_1 = P_1(X)$ and $X_2 = P_2(X)$, such that:*

$$f(z_1 + z_2\mathbf{i_2}) = f_{e1}(z_1 - z_2\mathbf{i_1})\mathbf{e_1} + f_{e2}(z_1 + z_2\mathbf{i_1})\mathbf{e_2} \ \forall \ z_1 + z_2\mathbf{i_2} \in X.$$

3. Bicomplex meromorphic functions

3.1. Basic definitions

In the complex plane, it is well known (see [24]) that a function f is meromorphic in an open set U if and only if f is a quotient g/h of two functions which are holomorphic in U where h is not identically zero in any component of U. Based on this definition we define a bicomplex meromorphic function as follows.

Definition 3.1. A function f is said to be bicomplex meromorphic in an open set $X \subset \mathbb{T}$ if f is a quotient g/h of two functions which are bicomplex holomorphic in X where h is not identically in the null-cone in any component of X.

Theorem 3.2. *Let $f : X \longrightarrow \mathbb{T}$ be a bicomplex meromorphic function on the open set $X \subset \mathbb{T}$. Then there exist meromorphic functions $f_{e1} : X_1 \longrightarrow \mathbb{C}(\mathbf{i_1})$ and $f_{e2} : X_2 \longrightarrow \mathbb{C}(\mathbf{i_1})$ with $X_1 = P_1(X)$ and $X_2 = P_2(X)$, such that:*

$$f(z_1 + z_2\mathbf{i_2}) = f_{e1}(z_1 - z_2\mathbf{i_1})\mathbf{e_1} + f_{e2}(z_1 + z_2\mathbf{i_1})\mathbf{e_2} \ \forall \ z_1 + z_2\mathbf{i_2} \in X.$$

Proof. Let $f : X \longrightarrow \mathbb{T}$ be a bicomplex meromorphic function on X. Then f is a quotient g/h of two functions which are bicomplex holomorphic in X where h is not identically in the null-cone in any component of X. Therefore, from Theorem 2.7 and Proposition 2.4, there exist holomorphic functions $g_{e1}, h_{e1} : X_1 \longrightarrow \mathbb{C}(\mathbf{i_1})$ and $g_{e2}, h_{e2} : X_2 \longrightarrow \mathbb{C}(\mathbf{i_1})$ with $X_1 = P_1(X)$ and $X_2 = P_2(X)$, such that:

$$g(z_1 + z_2\mathbf{i_2}) = g_{e1}(z_1 - z_2\mathbf{i_1})\mathbf{e_1} + g_{e2}(z_1 + z_2\mathbf{i_1})\mathbf{e_2}$$

and

$$h(z_1 + z_2\mathbf{i_2}) = h_{e1}(z_1 - z_2\mathbf{i_1})\mathbf{e_1} + h_{e2}(z_1 + z_2\mathbf{i_1})\mathbf{e_2}$$

$\forall\ z_1 + z_2\mathbf{i_2} \in X$ where h_{ei} is not identically zero in any component of X_i for $i = 1, 2$. Hence,

$$\begin{aligned} f(z_1 + z_2\mathbf{i_2}) &= \frac{g_{e1}(z_1 - z_2\mathbf{i_1})\mathbf{e_1} + g_{e2}(z_1 + z_2\mathbf{i_1})\mathbf{e_2}}{h_{e1}(z_1 - z_2\mathbf{i_1})\mathbf{e_1} + h_{e2}(z_1 + z_2\mathbf{i_1})\mathbf{e_2}} \\ &= \frac{g_{e1}(z_1 - z_2\mathbf{i_1})}{h_{e1}(z_1 - z_2\mathbf{i_1})}\mathbf{e_1} + \frac{g_{e2}(z_1 + z_2\mathbf{i_1})}{h_{e2}(z_1 + z_2\mathbf{i_1})}\mathbf{e_2} \\ &= f_{e1}(z_1 - z_2\mathbf{i_1})\mathbf{e_1} + f_{e2}(z_1 + z_2\mathbf{i_1})\mathbf{e_2} \end{aligned}$$

where f_{ei} is meromorphic in X_i for i=1,2. □

Definition 3.3. Let $f : X \longrightarrow \mathbb{T}$ be a bicomplex meromorphic function on the open set $X \subset \mathbb{T}$. We will say that $w = (z_1 - z_2\mathbf{i_1})\mathbf{e_1} + (z_1 + z_2\mathbf{i_1})\mathbf{e_2} \in X$ is a (strong) pole for the bicomplex meromorphic function

$$F(w) = F_{e1}(z_1 - z_2\mathbf{i_1})\mathbf{e_1} + F_{e1}(z_1 + z_2\mathbf{i_1})\mathbf{e_2}$$

if $z_1 - z_2\mathbf{i_1} \in P_1(X)$ (and) or $z_1 + z_2\mathbf{i_1} \in P_2(X)$ are poles for $F_{e1} : P_1(X) \longrightarrow \mathbb{C}(\mathbf{i_1})$ and $F_{e2} : P_2(X) \longrightarrow \mathbb{C}(\mathbf{i_1})$ respectively.

Remark 3.4. Poles of bicomplex meromorphic functions are not isolated singularities.

It is also easy to obtain the following characterization of poles.

Proposition 3.5. *Let $f : X \longrightarrow \mathbb{T}$ be a bicomplex meromorphic function on the open set $X \subset \mathbb{T}$. If $w_0 \in X$ then w_0 is a pole of f if and only if*

$$\lim_{w \to w_0} |f(w)| = \infty.$$

Definition 3.6. The order of a bicomplex meromorphic function

$$F(w) = F_{e1}(z_1 - z_2\mathbf{i_1})\mathbf{e_1} + F_{e2}(z_1 + z_2\mathbf{i_1})\mathbf{e_2}$$

is defined as

$$\rho(F) = max\{\rho(F_{e1}), \rho(F_{e2})\}.$$

Finally, to avoid any confusion, we will say that a function $f : \mathbb{T} \longrightarrow \mathbb{T}$ is a **transcendental bicomplex meromorphic function** on $\mathbb{T}$ if $f_{ei} : \mathbb{C}(\mathbf{i_1}) \longrightarrow \mathbb{C}(\mathbf{i_1})$ is a transcendental meromorphic function for $i = 1, 2$.

3.2. Factorization of bicomplex meromorphic functions

In this subsection we introduce the bicomplex version of the factorization of meromorphic functions in the plane.

Definition 3.7. Let F be a bicomplex meromorphic function on $\mathbb{T}$. Then F is said to have f and g as left and right factors respectively if F_{ei} has f_{ei} and g_{ei} as left and right factors respectively for $i = 1, 2$, i.e., f_{ei} is meromorphic and g_{ei} is entire (g_{ei} may be meromorphic when f_{ei} is rational) for $i = 1, 2$.

Remark 3.8. If F has f and g as left and right factors respectively then we always have the following factorization: $F(w) = f(g(w))$.

Proof. Let $F_{ei} = f_{ei}(g_{ei}(z))$ on $\mathbb{C}(\mathbf{i_1})$ for $i = 1, 2$. Then

$$\begin{aligned} f(g(w)) &= f(g_{e1}(z_1 - z_2\mathbf{i_1})\mathbf{e_1} + g_{e2}(z_1 + z_2\mathbf{i_1})\mathbf{e_2}) \\ &= f_{e1}(g_{e1}(z_1 - z_2\mathbf{i_1}))\mathbf{e_1} + f_{e2}(g_{e2}(z_1 + z_2\mathbf{i_1}))\mathbf{e_2}) \\ &= F_{e1}(z_1 - z_2\mathbf{i_1})\mathbf{e_1} + F_{e2}(z_1 + z_2\mathbf{i_1})\mathbf{e_2} \\ &= F(w). \end{aligned}$$

□

Theorem 3.9. *Let $F(w)$ be a bicomplex meromorphic function on $\mathbb{T}$. If $F(w) = f(g(w))$ where f is bicomplex meromorphic and g is bicomplex entire* (*g may be bicomplex meromorphic when f is bicomplex rational*) *then F has f and g as left and right factors respectively.*

Proof. From Theorem 3.2,

$$F(z_1 + z_2\mathbf{i_2}) = F_{e1}(z_1 - z_2\mathbf{i_1})\mathbf{e_1} + F_{e2}(z_1 + z_2\mathbf{i_1})\mathbf{e_2} \text{ on } \mathbb{T}$$

where F_{ei} is meromorphic on $\mathbb{C}(\mathbf{i_1})$ for $i = 1, 2$. Moreover, $\forall w \in \mathbb{T}$

$$\begin{aligned} F(w) &= f(g(w)) \\ &= f(g_{e1}(z_1 - z_2\mathbf{i_1})\mathbf{e_1} + g_{e2}(z_1 + z_2\mathbf{i_1})\mathbf{e_2}) \\ &= f_{e1}(g_{e1}(z_1 - z_2\mathbf{i_1}))\mathbf{e_1} + f_{e2}(g_{e2}(z_1 + z_2\mathbf{i_1}))\mathbf{e_2}. \end{aligned}$$

Hence, $F_{ei} = f_{ei}(g_{ei}(z))$ on $\mathbb{C}(\mathbf{i_1})$ where f_{ei} is meromorphic and g_{ei} is entire (g_{ei} may be meromorphic when f_{ei} is rational) for $i = 1, 2$. □

Proposition 3.10. *The converse of Theorem* 3.9 *is false.*

Proof. Supposed that F_{ei} has f_{ei} and g_{ei} as left and right factors respectively. In that case, the functions $F = f(g(w))$. However, in the situation where you have a rational function (with poles) for f_{e1} with a meromorphic function (with poles) for g_{e1} and an entire function for f_{e2} and g_{e2} then the complete function g will be bicomplex meromorphic (with poles) where f is not bicomplex rational. □

It is now possible to define the concept of prime (pseudo-prime) function in terms of the idempotent representation.

Definition 3.11. A bicomplex meromorphic function

$$F(z_1 + z_2\mathbf{i_2}) = F_{e1}(z_1 - z_2\mathbf{i_1})\mathbf{e_1} + F_{e2}(z_1 + z_2\mathbf{i_1})\mathbf{e_2} \text{ on } \mathbb{T}$$

is said prime (pseudo-prime), if the meromorphic functions F_{e1} and F_{e2} are prime (pseudo-prime).

Remark 3.12. All bicomplex polynomials are pseudo-prime, and bicomplex polynomials of prime degree are prime.

Theorem 3.13. *If every factorization of a bicomplex meromorphic function $F(w) = f(g(w))$ into left and right factors implies that f or g is bicomplex linear* (*bicomplex polynomial or f is rational*) *then F is prime* (*pseudo-prime*).

Proof. First, we note that a bicomplex meromorphic function $h(z_1 + z_2\mathbf{i_2}) = h_{e1}(z_1 - z_2\mathbf{i_1})\mathbf{e_1} + h_{e2}(z_1 + z_2\mathbf{i_1})\mathbf{e_2}$ is bicomplex linear (bicomplex polynomial) if and only if h_{ei} is linear (polynomial) for $i = 1, 2$. Now, since every factorization of $F(w)$ of the form $f(g(w))$ implies that either f or g is bicomplex linear (bicomplex polynomial or f is bicomplex rational), then f_{ei} or g_{ei} is linear (polynomial or f_{ei} is rational) for $i = 1, 2$. This further implies that F_{ei} is prime (pseudo-prime) for $i = 1, 2$. □

Proposition 3.14. *The converse of Theorem* 3.13 *is false.*

Proof. Supposed that every factorization of $F_{ei}(w) = f_{ei}(g_{ei}(w))$ into left and right factors implies that f_{ei} or g_{ei} is polynomial or f_{ei} is rational for $i = 1, 2$. In that case, the function F is supposed to be pseudo-prime but in the situation where you have a polynomial for f_{e1} and a rational function for f_{e2} then the complete function f in the bicomplex space will be neither a bicomplexe polynomial nor a bicomplex rational function. □

In 1973, Gross, Osgoods and Yang posed the following problem (see [6]): Given any transcendental entire function f, does there exist a meromorphic function g such that fg is prime? In [13], Noda gave an affirmative answer to the above problem and in [8] Qiao and Yongxing extended this to meromorphic functions. The next theorem will show that the same result is also true in the bicomplex case.

Theorem 3.15. *Let F be any bicomplex transcendental meromorphic function, then there exists a bicomplex meromorphic function G such that GF is prime.*

Proof. Let $F(w) = F_{e1}(z_1 - z_2\mathbf{i_1})\mathbf{e_1} + F_{e1}(z_1 + z_2\mathbf{i_1})\mathbf{e_2}$ be any bicomplex transcendental meromorphic function. Therefore, F_{ei} is a transcendental meromorphic function for $i = 1, 2$. Hence, there exists a meromorphic function G_{ei} such that $F_{ei}G_{ei}$ is prime for $i = 1, 2$ (see Qiao and Yongxing [8]). Now, from Definition 3.11, FG is prime when G is defined as follows:

$$G(w) := G_{e1}(z_1 - z_2\mathbf{i_1})\mathbf{e_1} + G_{e1}(z_1 + z_2\mathbf{i_1})\mathbf{e_2}. \qquad \square$$

Theorem 3.16. *A bicomplex transcendental entire function of finite order such that $F(\mathbb{T}) \subset \mathbb{T}^{-1}$ is pseudo-prime.*

Proof. Let F be a bicomplex transcendental entire function such that $F(\mathbb{T}) \subset \mathbb{T}^{-1}$. Since $\rho(F)$ is finite, $\rho(F_{ei})$ is also finite for $i = 1, 2$. Moreover, since $F(\mathbb{T})$ is always invertible, it follows from Proposition 2.4 that the entire function F_{ei} has no zeros for $i = 1, 2$. Thus by a result of Gross (see [5], p. 215, Theorem 1) it follows that each F_{ei} is pseudo-prime. Hence by Definition 3.11, F is pseudo-prime. □

Example. $\exp(z_1 + z_2\mathbf{i_2})$ is pseudo-prime.

In [5] Fred Gross conjectured that if f and g are non-linear entire functions, at least one of them transcendental, then the composite function $f \circ g$ has infinitely many fix-points. Its factorization version is: if P is a polynomial and if α is a non-constant entire function, then the function $F(z) = P(z)\exp(\alpha(z)) + z$ is prime. Bergweiler [2] proved this long pending conjecture in its general form as: if P and Q are polynomials and α is an entire function such that Q and α are non-constant, and P, Q and α do not have a non-linear common right factor, then $P(z)\exp(\alpha(z)) + Q(z)$ is prime, and conversely also. The bicomplex analogue of Bergweiler's result also holds with stronger hypotheses.

Definition 3.17. Let $F(w) = F_{e1}(z_1 - z_2\mathbf{i_1})\mathbf{e_1} + F_{e1}(z_1 + z_2\mathbf{i_1})\mathbf{e_2} : D \longrightarrow \mathbb{T}$ be any bicomplex function. The function F is said to be strongly non-constant (non-linear) on D if F_{ei} is non-constant (non-linear) on $P_i(D)$ for $i = 1, 2$.

Theorem 3.18. *Let P and Q be bicomplex polynomials and α be a bicomplex entire function. Suppose that Q and α are strongly non-constant for $i = 1, 2$ and let $F(w) = P(w)\exp(\alpha(w)) + Q(w)$. Then F is prime if and only if P, Q, and α do not have strongly non-linear bicomplex common right factor.*

Proof. Let

$$\begin{aligned} P(z_1 + z_2\mathbf{i_2}) &= P_{e1}(z_1 - z_2\mathbf{i_1})\mathbf{e_1} + P_{e2}(z_1 + z_2\mathbf{i_1})\mathbf{e_2} \\ Q(z_1 + z_2\mathbf{i_2}) &= Q_{e1}(z_1 - z_2\mathbf{i_1})\mathbf{e_1} + Q_{e2}(z_1 + z_2\mathbf{i_1})\mathbf{e_2} \\ \alpha(z_1 + z_2\mathbf{i_2}) &= \alpha_{e1}(z_1 - z_2\mathbf{i_1})\mathbf{e_1} + \alpha_{e2}(z_1 + z_2\mathbf{i_1})\mathbf{e_2}. \end{aligned}$$

Then it follows that

$$g(z_1 + z_2\mathbf{i_2}) = g_{e1}(z_1 - z_2\mathbf{i_1})\mathbf{e_1} + g_{e2}(z_1 + z_2\mathbf{i_1})$$

is a strongly non-linear bicomplex common right factor of P, Q, and α if and only if g_{ei} is a non-linear common right factor of P_{ei}, Q_{ei}, and α_{ei} for $i = 1, 2$.

Now since

$$f \circ g = (f_{e1} \circ g_{e1})\mathbf{e_1} + (f_{e2} \circ g_{e2})\mathbf{e_2}$$

and

$$f + g = (f_{e1} + g_{e1})\mathbf{e_1} + (f_{e2} + g_{e2})\mathbf{e_2},$$

we have

$$\begin{aligned}F(w) &= P(w)\exp(\alpha(w)) + Q(w)\\ &= (P_{e1}(w_1)\mathbf{e_1} + P_{e2}(w_2)\mathbf{e_2})(e^{\alpha_{e1}(w_1)}\mathbf{e_1} + e^{\alpha_{e2}(w_2)}\mathbf{e_2})\\ &\quad + (Q_{e1}(w_1)\mathbf{e_1} + Q_{e2}(w_2)\mathbf{e_2})\\ &= (P_{e1}(w_1)e^{\alpha_{e1}(w_1)} + Q_{e1}(w_1))\mathbf{e_1} + (P_{e2}(w_2)e^{\alpha_{e2}(w_2)} + Q_{e2}(w_2))\mathbf{e_2}\end{aligned}$$

where $w = z_1 + z_2\mathbf{i_2}$, $w_1 = z_1 - z_2\mathbf{i_1}$ and $w_2 = z_1 + z_2\mathbf{i_1}$. Writing $F(w) = F_{e1}(w_1)\mathbf{e_1} + F_{e2}(w_2)\mathbf{e_2}$ we find from above that

$$F_{ei}(w_i) = P_{ei}(w_i)e^{\alpha_{ei}(w_i)} + Q_{ei}(w_i)$$

which by Bergweiler's result (see [2]) is prime if and only if P_{ei}, Q_{ei}, and α_{ei} do not have a non-linear common right factor for $i = 1, 2$. Therefore, since an arbitrary bicomplex function $f(w)$ is strongly non-constant if and only if $P_1(f(w))$ and $P_2(f(w))$ is non-constant, we obtain from our hypotheses and Theorem 3.13 that $F(w) = P(w)\exp(\alpha(w)) + Q(w)$ is prime if and only if P, Q, and α do not have strongly non-linear bicomplex common right factor. □

Remark 3.19. As for one complex variable, the Theorem 3.18 implies that if f and g are strongly non-linear bicomplex entire functions, at least one of them transcendental, then the composite function $f \circ g$ has infinitely many fix-points.

In [1], Baoqin and Guodong proved the following: if f is any transcendental meromorphic function, then for any non-constant fractional linear function ϕ, the set

$$\{a \in \mathbb{C} : f(z) + a\phi(z) \text{ is not prime}\}$$

is at most countable. We will now show that bicomplex version of this result is false.

Theorem 3.20. *Let f be any bicomplex transcendental meromorphic function, then for any non-constant fractional linear bicomplex function ϕ, the set*

$$\{a \in \mathbb{T} : f(w) + a\phi(w) \ \textit{is not prime}\}$$

is empty or of cardinality $\aleph_1$.

Proof. For $w = z_1 + z_2\mathbf{i_2}$, $w_1 = z_1 - z_2\mathbf{i_1}$, and $w_2 = z_1 + z_2\mathbf{i_1}$ writing

$$f(w) = f_{e1}(w_1)\mathbf{e_1} + f_{e2}(w_2)\mathbf{e_2}$$

$$\phi(w) = \phi_{e1}(w_1)\mathbf{e_1} + \phi_{e2}(w_2)\mathbf{e_2},$$

where f_{ei} is transcendental meromorphic and ϕ_{ei} is fractional linear for $i = 1, 2$ where ϕ_{ei} is non-constant for $i = 1$ or $i = 2$. Without loss of generality, let ϕ_{e1} be non-constant. Then, the set

$$\{\alpha \in \mathbb{C}(\mathbf{i}_1) : f_{e1}(z) + \alpha\phi_{e1}(z) \text{ is not prime}\}$$

is at most countable.

Now by taking $a = a_{e1}\mathbf{e_1} + a_{e2}\mathbf{e_2}$ and since

$$f(w) + a\phi(w) = (f_{e1}(w_1) + a_{e1}\phi_{e1}(w_1))\mathbf{e_1} + (f_{e2}(w_2) + a_{e2}\phi_{e2}(w_2))\mathbf{e_2},$$

it follows from Definition 3.11 that:

$$\begin{aligned} f(w) + a\phi(w) \text{ is not prime } &\Leftrightarrow \quad f_{e1}(z) + a_{e1}\phi_{e1}(z) \text{ is not prime} \\ &\ \ \textbf{or} \quad f_{e2}(z) + a_{e2}\phi_{e2}(z) \text{ is not prime.} \end{aligned}$$

Since $|\{\alpha \in \mathbb{C}(\mathbf{i_1}) : f_{e2}(z) + \alpha\phi_{e2}(z) \text{ is prime or not}\}| = |\mathbb{C}(\mathbf{i_1})| = \aleph_1$ then

$$\{a \in \mathbb{T} : f(w) + a\phi(w) \text{ is not prime}\}$$

is also of cardinality $\aleph_1$ except if the set is empty. □

Finally, by using idempotent representations of bicomplex numbers and bicomplex functions, and by using Definition 3.11 and Theorem 4.11 in [3] we have the following result.

Theorem 3.21. *Every bicomplex meromorphic solution of an nth-order ordinary bicomplex differential equation with bicomplex rational functions as coefficients is pseudo-prime.*

Example. $\cos(z_1 + z_2\mathbf{i_2})$ is pseudo-prime since it satisfies the ordinary bicomplex linear differential equation $y''(w) - y(w) = 0$.

Theorem 3.22. *Let $F(z_1 + z_2\mathbf{i_2}) = \cos(z_1 + z_2\mathbf{i_2})$, the possible forms of the factorization of $F = f \circ g$ are as follows:*

$$f(\zeta_1 + \zeta_2\mathbf{i_2}) = f_{e1}(\zeta_1 - \zeta_2\mathbf{i_1})\mathbf{e_1} + f_{e2}(\zeta_1 + \zeta_2\mathbf{i_1})\mathbf{e_2}$$

and

$$g(z_1 + z_2\mathbf{i_2}) = g_{e1}(z_1 - z_2\mathbf{i_1})\mathbf{e_1} + g_{e2}(z_1 + z_2\mathbf{i_1})\mathbf{e_2}$$

where the couple $(f_{ei}(z),\ g_{ei}(z))$ is chosen from the following possible forms of factorization of $\cos z$ in the complex plane:

(i) $f_{ei}(z) = \cos\sqrt{z}$, $g_{ei}(z) = z^2$;
(ii) $f_{ei}(z) = T_n(z)$, $g_{ei}(z) = \cos\frac{z}{n}$, *where $T_n(z)$ denotes the nth Chebyshev polynomial ($n \geq 2$);*
(iii) $f_{ei}(z) = \frac{1}{2}(z^{-n} + z^n)$, $g_{ei}(z) = e^{\frac{\mathbf{i}z}{n}}$, *where n denotes a non-negative integer*

for $i = 1, 2$. Moreover, if the couple $(f_{ei}(z),\ g_{ei}(z))$ is of the form (i) *or* (ii) *for each $i = 1$ and 2, then f and g are, in particular, entire holomorphic mappings of two complex variables.*

Proof. The proof is a direct consequence of the Theorems 2.2, 2.6 and 3.9 used with the three possible forms of factorization of $\cos z$ in the complex plane (see [3]). □

Corollary 3.23. *The complex cosine $\cos z$ has three possible forms of factorization in the complex plane and six additional possible forms of factorization in the bicomplex space.*

Proof. From Theorem 3.22 we have nine possible forms of factorization of the bicomplex cosine: $\cos(z_1 + z_2\mathbf{i_2})$. If we put $z_2 = 0$, we obtain automatically nine possible forms of factorization of the cosine in the complex plane. However, when $f(\zeta_1+\zeta_2\mathbf{i_2}) = f_{e1}(\zeta_1-\zeta_2\mathbf{i_1})\mathbf{e_1}+f_{e1}(\zeta_1+\zeta_2\mathbf{i_1})\mathbf{e_2}$ and $g(z_1+z_2\mathbf{i_2}) = g_{e1}(z_1-z_2\mathbf{i_1})\mathbf{e_1}+g_{e1}(z_1+z_2\mathbf{i_1})\mathbf{e_2}$ with $z_2 = 0$, we have that $g(z_1) = g_{e1}(z_1)\mathbf{e_1} + g_{e1}(z_1)\mathbf{e_2} = g_{e1}(z_1)$ and $(f \circ g)(z_1) = (f_{e1} \circ g_{e1})(z_1)$. In that case, we come back to the three classical complex forms of factorization of $\cos z_1$. Hence, the complex cosine has exactly six new possible forms of factorization in the bicomplex space. □

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