Bicomplex Functional Analysis

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- Bicomplex numbers, just like quaternions, are a generalization of complex numbers by means of entities specified by four real numbers. These two number systems, however, are different in two important ways: quaternions, which form a division algebra, are noncommutative, whereas bicomplex numbers are commutative but do not form a division algebra.
- Division algebras do not have zero divisors, that is, nonzero elements whose product is zero. Many believe that any attempt to generalize quantum mechanics to number systems other than complex numbers should retain the division algebra property. Indeed considerable work has been done over the years on quaternionic quantum mechanics.

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- In the past few years, however, it was pointed out that several features of quantum mechanics can be generalized to bicomplex numbers. A generalization of Schrödinger's equation for a particle in one dimension was proposed, and self-adjoint operators were defined on finite-dimensional bicomplex Hilbert spaces. Recently, eigenvalues and eigenfunctions of the bicomplex analogue of the quantum harmonic oscillator Hamiltonian were obtained in full generality.
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- The perspective of generalizing quantum mechanics to bicomplex numbers motivates us in developing further mathematical tools related to infinite-dimensional bicomplex Hilbert spaces and operators acting on them.

Definition

Bicomplex numbers are defined as

$$\mathbb{M}(2) := \{z_1 + z_2\mathbf{i_2} \mid z_1, z_2 \in \mathbb{C}(\mathbf{i_1})\}$$

where the imaginary units $\mathbf{i_1}, \mathbf{i_2}$ and $\mathbf{j}$ are governed by the rules: $\mathbf{i_1}^2 = \mathbf{i_2}^2 = -1$, $\mathbf{j}^2 = 1$ and

$$\begin{array}{rcccl} \mathbf{i_1}\mathbf{i_2} & = & \mathbf{i_2}\mathbf{i_1} & = & \mathbf{j}, \\ \mathbf{i_1}\mathbf{j} & = & \mathbf{j}\mathbf{i_1} & = & -\mathbf{i_2}, \\ \mathbf{i_2}\mathbf{j} & = & \mathbf{j}\mathbf{i_2} & = & -\mathbf{i_1}. \end{array}$$

- Note that we define $\mathbb{C}(\mathbf{i}_k) := \{x + y\mathbf{i}_k \mid \mathbf{i}_k^2 = -1 \text{ and } x, y \in \mathbb{R}\}$ for $k = 1, 2$.

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In fact, the bicomplex numbers

$$\mathbb{M}(2) \cong \mathrm{Cl}_{\mathbb{C}}(1,0) \cong \mathrm{Cl}_{\mathbb{C}}(0,1)$$

are *unique* among the complex Clifford algebras in the sense that they are commutative but not division algebra. It is also convenient to write the set of bicomplex numbers as

$$\mathbb{M}(2) := \{w_0 + w_1\mathbf{i_1} + w_2\mathbf{i_2} + w_3\mathbf{j} \mid w_0, w_1, w_2, w_3 \in \mathbb{R}\}.$$

- In particular, if we put $z_1 = x$ and $z_2 = y\mathbf{i_1}$ with $x, y \in \mathbb{R}$ in $z_1 + z_2\mathbf{i_2}$, then we obtain the following subalgebra of hyperbolic numbers, also called duplex numbers:

$$\mathbb{D} := \{x + y\mathbf{j} \mid \mathbf{j}^2 = 1,\ x, y \in \mathbb{R}\} \cong \mathrm{Cl}_{\mathbb{R}}(0,1).$$

- Zero divisors make up the so-called null cone $\mathcal{NC}$. That terminology comes from the fact that when w is written as $z_1 + z_2\mathbf{i_2}$, zero divisors are such that $z_1^2 + z_2^2 = 0$.

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- Complex conjugation plays an important role both for algebraic and geometric properties of $\mathbb{C}$. For bicomplex numbers, there are three possible conjugations. Let $w \in \mathbb{M}(2)$ and $z_1, z_2 \in \mathbb{C}(\mathbf{i_1})$ such that $w = z_1 + z_2\mathbf{i_2}$. Then we define the three conjugations as:

$$w^{\dagger_1} = (z_1 + z_2\mathbf{i_2})^{\dagger_1} := \overline{z}_1 + \overline{z}_2\mathbf{i_2},$$
$$w^{\dagger_2} = (z_1 + z_2\mathbf{i_2})^{\dagger_2} := z_1 - z_2\mathbf{i_2},$$
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where $\overline{z}_k$ is the standard complex conjugate of complex numbers $z_k \in \mathbb{C}(\mathbf{i_1})$.

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We know that the product of a standard complex number with its conjugate gives the square of the Euclidean metric in $\mathbb{R}^2$. The analogs of this, for bicomplex numbers, are the following. Let $z_1, z_2 \in \mathbb{C}(\mathbf{i_1})$ and $w = z_1 + z_2\mathbf{i_2} \in \mathbb{M}(2)$, then we have that:

$$|w|^2_{\mathbf{i_1}} := w \cdot w^{\dagger_2} \in \mathbb{C}(\mathbf{i_1}),$$

$$|w|^2_{\mathbf{i_2}} := w \cdot w^{\dagger_1} \in \mathbb{C}(\mathbf{i_2}),$$

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In this talk we will often use the Euclidean $\mathbb{R}^4$-norm defined as

$$|w| := \sqrt{|z_1|^2 + |z_2|^2} = \sqrt{\mathrm{Re}(|w|^2_{\mathbf{j}})}\,.$$

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- It is also important to know that every bicomplex number $w = z_1 + z_2\mathbf{i_2}$ has the following unique idempotent representation:

$$z_1 + z_2\mathbf{i_2} = (z_1 - z_2\mathbf{i_1})\mathbf{e_1} + (z_1 + z_2\mathbf{i_1})\mathbf{e_2}.$$

where $\mathbf{e_1} = \frac{1+\mathbf{j}}{2}$ and $\mathbf{e_2} = \frac{1-\mathbf{j}}{2}$.

- From this, we can introduce two projection operators

$$P_1 : (z_1 + z_2\mathbf{i_2}) \in \mathbb{M}(2) \mapsto (z_1 + z_2\mathbf{i_2})_{\hat{1}} \in \mathbb{C}(\mathbf{i_1}),$$
$$P_2 : (z_1 + z_2\mathbf{i_2}) \in \mathbb{M}(2) \mapsto (z_1 + z_2\mathbf{i_2})_{\hat{2}} \in \mathbb{C}(\mathbf{i_1}).$$

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- Bicomplex numbers make up a commutative ring. What vector spaces are to fields, modules are to rings. A module defined over the ring $\mathbb{M}(2)$ of bicomplex numbers will be called an $\mathbb{M}(2)$-*module*.

Definition

Let M be an $\mathbb{M}(2)$-module. For $k = 1, 2$, we define V_k as the set of all elements of the form $\mathbf{e_k}|\psi\rangle$, with $|\psi\rangle \in M$. Succinctly, $V_1 := \mathbf{e_1}M$ and $V_2 := \mathbf{e_2}M$.

- We have used Dirac's notation for elements of M which, following, we will call *kets*.
- For $k = 1, 2$, addition and multiplication by a $\mathbb{C}(\mathbf{i_1})$ scalar are closed in V_k. Therefore, V_k is a vector space over $\mathbb{C}(\mathbf{i_1})$.

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Theorem

The $\mathbb{M}(2)$-module M can be viewed as a vector space M' over $\mathbb{C}(\mathbf{i_1})$, and $M' = V_1 \oplus V_2$.

- Henceforth we will write $|\psi_{\mathbf{k}}\rangle = \mathbf{e_k}|\psi\rangle$, keeping in mind that $\mathbf{e_k}|\psi_{\mathbf{k}}\rangle = |\psi_{\mathbf{k}}\rangle \in V_k$ for $k = 1, 2$.
- From a set-theoretical point of view, M and M' are identical. In this sense we can say, perhaps improperly, that the **module** M can be decomposed into the direct sum of two vector spaces over $\mathbb{C}(\mathbf{i_1})$, i.e. $M = V_1 \oplus V_2$.

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- The norm of a vector is an important concept in vector space theory. We will now generalize it to $\mathbb{M}(2)$-modules, making use of the association established in the last Theorem.

definition

Let M be an $\mathbb{M}(2)$-module and let M' be the associated vector space. We say that $\|\cdot\| : M \longrightarrow \mathbb{R}$ is a $\mathbb{M}(2)$-**norm** on M if the following holds:

1. $\|\cdot\| : M' \longrightarrow \mathbb{R}$ is a norm;
2. $\big\|w \cdot |\psi\rangle\big\| \leq \sqrt{2}|w| \cdot \big\||\psi\rangle\big\|$, $\forall w \in \mathbb{M}(2)$, $\forall|\psi\rangle \in M$.

- A $\mathbb{M}(2)$-module with a $\mathbb{M}(2)$-**norm** is called a **normed** $\mathbb{M}(2)$-**module**.

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- In vector space theory, a norm can be induced by a scalar product. Having in mind the use of such norms, we will used the following definition of a **bicomplex scalar product** (the physicists' ordering convention being used).

Definition

Let M be an $\mathbb{M}(2)$-module. Suppose that with each pair $|\psi\rangle$ and $|\phi\rangle$ in M, taken in this order, we associate a bicomplex number $(|\psi\rangle, |\phi\rangle)$. We say that the association defines a bicomplex scalar (or inner) product if it satisfies the following conditions:

1. $(|\psi\rangle, |\phi\rangle + |\chi\rangle) = (|\psi\rangle, |\phi\rangle) + (|\psi\rangle, |\chi\rangle)$, $\forall |\psi\rangle, |\phi\rangle, |\chi\rangle \in M$;
2. $(|\psi\rangle, \alpha|\phi\rangle) = \alpha(|\psi\rangle, |\phi\rangle)$, $\forall \alpha \in \mathbb{M}(2)$, $\forall |\psi\rangle, |\phi\rangle \in M$;
3. $(|\psi\rangle, |\phi\rangle) = (|\phi\rangle, |\psi\rangle)^{\dagger_3}$, $\forall |\psi\rangle, |\phi\rangle \in M$;
4. $(|\psi\rangle, |\psi\rangle) = 0 \;\Leftrightarrow\; |\psi\rangle = 0$, $\forall |\psi\rangle \in M$.

OPEN QUESTION:

How to construct the theory if we define the bicomplex scalar product with the other conjugate $\dagger_2$ (or $\dagger_1$) ?

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- Property 3 implies that $(|\psi\rangle, |\psi\rangle) \in \mathbb{D}$. Definition 3 is intended to be very general. In this paper we shall be more restrictive, by requiring the bicomplex scalar product $(\cdot, \cdot)$ to be *hyperbolic positive*, that is,

$$(|\psi\rangle, |\psi\rangle) \in \mathbb{D}_+ := \{\alpha \mathbf{e_1} + \beta \mathbf{e_2} | \alpha, \beta \geq 0\}, \ \forall |\psi\rangle \in M.$$

- From the last definition it is easy to see that the following projection of a bicomplex scalar product:

$$(\cdot, \cdot)_{\widehat{k}} := P_k((\cdot, \cdot)) : M \times M \longrightarrow \mathbb{C}(\mathbf{i_1})$$

is a **standard scalar product** on V_k, for $k = 1, 2$.

Theorem

Let $|\psi\rangle, |\phi\rangle \in M$, then

$$(|\psi\rangle, |\phi\rangle) = \mathbf{e_1}(|\psi_1\rangle, |\phi_1\rangle)_{\widehat{1}} + \mathbf{e_2}(|\psi_2\rangle, |\phi_2\rangle)_{\widehat{2}}.$$

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- Property 3 implies that $(|\psi\rangle, |\psi\rangle) \in \mathbb{D}$. Definition 3 is intended to be very general. In this paper we shall be more restrictive, by requiring the bicomplex scalar product $(\cdot,\cdot)$ to be *hyperbolic positive*, that is,

$$(|\psi\rangle, |\psi\rangle) \in \mathbb{D}_+ := \{\alpha \mathbf{e_1} + \beta \mathbf{e_2} | \alpha, \beta \geq 0\}, \ \forall |\psi\rangle \in M.$$

- From the last definition it is easy to see that the following projection of a bicomplex scalar product:

$$(\cdot,\cdot)_{\widehat{k}} := P_k((\cdot,\cdot)) : M \times M \longrightarrow \mathbb{C}(\mathbf{i_1})$$

is a **standard scalar product** on V_k, for $k = 1, 2$.

Theorem

Let $|\psi\rangle, |\phi\rangle \in M$, then

$$(|\psi\rangle, |\phi\rangle) = \mathbf{e_1}(|\psi_\mathbf{1}\rangle, |\phi_\mathbf{1}\rangle)_{\widehat{1}} + \mathbf{e_2}(|\psi_\mathbf{2}\rangle, |\phi_\mathbf{2}\rangle)_{\widehat{2}}.$$

Moreover, the bicomplex scalar product is **completely characterized** by the two standard scalar products $(\cdot,\cdot)_{\widehat{k}}$ on V_k.

Definition

Let M be an $\mathbb{M}(2)$-module and let $(\cdot,\cdot)$ be a bicomplex scalar product defined on M. The space $\{M,(\cdot,\cdot)\}$ is called a $\mathbb{M}(2)$-inner product space, or bicomplex pre-Hilbert space.

- If V_1 and V_2 are complete, then $M' = V_1 \oplus V_2$ is a direct sum of two Hilbert spaces. It is easy to see that M' is also a Hilbert space, when the following natural scalar product is defined over the direct sum:

$$(|\psi_1\rangle \oplus |\psi_2\rangle, |\phi_1\rangle \oplus |\phi_2\rangle) = (|\psi_1\rangle, |\phi_1\rangle)_{\widehat{1}} + (|\psi_2\rangle, |\phi_2\rangle)_{\widehat{2}}.$$

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- From this scalar product, we can define a **norm** on the vector space M':

$$\big|\big||\phi\rangle\big|\big| := \frac{1}{\sqrt{2}}\sqrt{(|\phi_{\mathbf{1}}\rangle, |\phi_{\mathbf{1}}\rangle)_{\widehat{1}} + (|\phi_{\mathbf{2}}\rangle, |\phi_{\mathbf{2}}\rangle)_{\widehat{2}}}$$
$$= \frac{1}{\sqrt{2}}\sqrt{\big||\phi_{\mathbf{1}}\rangle\big|_1^2 + \big||\phi_{\mathbf{2}}\rangle\big|_2^2}\,. \qquad (1)$$

Here we wrote

$$\big||\phi_{\mathbf{k}}\rangle\big|_k = \sqrt{(|\phi_{\mathbf{k}}\rangle, |\phi_{\mathbf{k}}\rangle)_{\widehat{k}}}\,,$$

where $|\cdot|_k$ is the natural scalar product induced norm on V_k. The $1/\sqrt{2}$ factor in (1) is introduced so as to relate in a simple manner the norm with the bicomplex scalar product.

- Indeed we have

$$\big|\big||\phi\rangle\big|\big| = \frac{1}{\sqrt{2}}\sqrt{(|\phi_1\rangle, |\phi_1\rangle)_{\hat{1}} + (|\phi_2\rangle, |\phi_2\rangle)_{\hat{2}}} = \left|\sqrt{(|\phi\rangle, |\phi\rangle)}\right|.$$

- Hence, since $(\cdot,\cdot) \in \mathbb{D}_+$, we have in general that

$$(|\phi\rangle, |\phi\rangle) \neq \big|\big||\phi\rangle\big|\big|^2 \in \mathbb{R}_+$$

except when $(|\phi_1\rangle, |\phi_1\rangle)_{\hat{1}} = (|\phi_2\rangle, |\phi_2\rangle)_{\hat{2}}$.

- It is easy to check that $\|\cdot\|$ is a $\mathbb{M}(2)$-**norm** on M and that the $\mathbb{M}(2)$-module M is **complete** with respect to the following metric on M:

$$d(|\phi\rangle, |\psi\rangle) = \big|\big||\phi\rangle - |\psi\rangle\big|\big|.$$

Thus M is a **complete** $\mathbb{M}(2)$-**module**.

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- Let us summarize what we found by means of a definition, an example and a theorem.

Definition

A **bicomplex Hilbert space** is a $\mathbb{M}(2)$-inner product space M which is complete with respect to the induced $\mathbb{M}(2)$-norm (1).

Example

Consider this following class of bicomplex functions with $\mu \in \mathbb{R}^q$.

$$f(\mu) = f_{\hat{1}}(\mu)\,\mathbf{e_1} + f_{\hat{2}}(\mu)\,\mathbf{e_2}. \tag{2}$$

We say that f is a bicomplex square-integrable function if and only if the $f_{\hat{s}}$ are both square-integrable functions, that is,

$$\int |f_{\hat{s}}(\mu)|^2 \,\mathrm{d}\mu < \infty \tag{3}$$

for $s = 1$ and 2. Here $\mathrm{d}\mu$ is the Lebesgue mesure on $\mathbb{R}^q$.

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Example

We denote by $\mathcal{F}_q$ the set of bicomplex square-integrable functions of q real variables. It can be shown that with standard addition and multiplication, $\mathcal{F}_q$ makes up a $\mathbb{M}(2)$-module. This module is explicitly denoted as $(\mathcal{F}_q, \mathbb{M}(2), +, \cdot)$, and it obviously has infinite dimensions. For any $f, g \in \mathcal{F}_q$, the following binary mapping takes two bicomplex square-integrable functions and transforms them into a unique bicomplex number:

$$(f, g) := \int f^{\dagger_3}(\boldsymbol{\mu})\, g(\boldsymbol{\mu})\, \mathrm{d}\boldsymbol{\mu} = \sum_s \mathbf{e}_s \int \overline{f_{\widehat{s}}(\boldsymbol{\mu})} g_{\widehat{s}}(\boldsymbol{\mu})\, \mathrm{d}\boldsymbol{\mu}. \tag{4}$$

If we identify functions that differ only on a set of measure zero, the binary mapping (4) satisfies all the properties of a scalar product.

Example

Explicitly,

1. $(f, g + h) = (f, g) + (f, h)$;
2. $(f, \alpha g) = \alpha(f, g)$;
3. $(f, g) = (g, h)^{\dagger_3}$;
4. $(f, f) = 0$ if and only if $f = 0$.

The functions f and g are orthogonal if their scalar product vanish. We say that f is normalized if $(f, f) = 1$. With (4), one can define an induced $\mathbb{M}(2)$-norm on $\mathcal{F}_q$ as

$$\|f\| := \frac{1}{\sqrt{2}}\sqrt{(f,f)_{\widehat{1}} + (f,f)_{\widehat{2}}} = \frac{1}{\sqrt{2}}\sqrt{\sum_s \int |f_{\widehat{s}}(\boldsymbol{\mu})|^2 \, \mathrm{d}\boldsymbol{\mu}}. \qquad (5)$$

With this induced $\mathbb{M}(2)$-norm on $\mathcal{F}_q$ the structure

$(\mathcal{F}_q, \mathbb{M}(2), +, \cdot, (\ ,\ ), \|\ \ \|)$ is a **bicomplex Hilbert space**.

Theorem

Let M be a bicomplex Hilbert space. Then $(V_k, (\cdot,\cdot)_{\widehat{k}})$ is a complex (in $\mathbb{C}(\mathbf{i_1})$) Hilbert space for $k = 1, 2$.

- As a direct application of this result, we obtain the following **Bicomplex Riesz Representation Theorem**.

Theorem (Riesz)

Let $\{M, (\cdot,\cdot)\}$ be a bicomplex Hilbert space and let $f : M \to \mathbb{M}(2)$ be a continuous linear functional on M. Then there is a unique $|\psi\rangle \in M$ such that $\forall |\phi\rangle \in M$, $f(|\phi\rangle) = (|\psi\rangle, |\phi\rangle)$.

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- We will now use the Dirac notation for the scalar product:

$$(|\psi\rangle, |\phi\rangle) = \langle\psi|\phi\rangle\,.$$

The one-to-one correspondence between *bra* $\langle\cdot|$ and *ket* $|\cdot\rangle$ can be established from the **Bicomplex Riesz Representation Theorem** using

$$f(|\phi\rangle) := \langle\psi|(|\phi\rangle) = \langle\psi|\phi\rangle\,.$$

- One can easily show that

Corollary

$$\langle\psi|\phi\rangle = \mathbf{e_1}\,\langle\psi_1|\phi_1\rangle_{\widehat{1}} + \mathbf{e_2}\,\langle\psi_2|\phi_2\rangle_{\widehat{2}}\,.$$

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- We close this section by showing a general version of Schwarz's inequality in a bicomplex Hilbert space.

Theorem (Bicomplex Schwarz inequality)

Let $|\psi\rangle, |\phi\rangle \in M$. Then

$$|\langle\psi|\phi\rangle| \leq \sqrt{2}\,\big\||\psi\rangle\big\|\ \big\||\phi\rangle\big\|.$$

Proof.

From the complex (in $\mathbb{C}(\mathbf{i_1})$) Schwarz inequality we have

$$|\langle\psi_{\mathbf{k}}|\phi_{\mathbf{k}}\rangle_{\widehat{k}}|^2 \leq \big||\psi_{\mathbf{k}}\rangle\big|_k^2 \cdot \big||\phi_{\mathbf{k}}\rangle\big|_k^2, \quad \forall|\psi_{\mathbf{k}}\rangle, |\phi_{\mathbf{k}}\rangle \in V_k.$$

Therefore, if $|\psi\rangle, |\phi\rangle \in M$, we obtain that

$$\begin{aligned}
|\langle\psi|\phi\rangle| &= |\mathbf{e_1}\langle\psi_{\mathbf{1}}|\phi_{\mathbf{1}}\rangle_{\widehat{1}} + \mathbf{e_2}\langle\psi_{\mathbf{2}}|\phi_{\mathbf{2}}\rangle_{\widehat{2}}| \\
&= \frac{1}{\sqrt{2}}\sqrt{|\langle\psi_{\mathbf{1}}|\phi_{\mathbf{1}}\rangle_{\widehat{1}}|^2 + |\langle\psi_{\mathbf{2}}|\phi_{\mathbf{2}}\rangle_{\widehat{2}}|^2} \\
&\leq \frac{1}{\sqrt{2}}\sqrt{\big||\psi_{\mathbf{1}}\rangle\big|_1^2 \cdot \big||\phi_{\mathbf{1}}\rangle\big|_1^2 + \big||\psi_{\mathbf{2}}\rangle\big|_2^2 \cdot \big||\phi_{\mathbf{2}}\rangle\big|_2^2} \\
&\leq \frac{1}{\sqrt{2}}\sqrt{(\big||\psi_{\mathbf{1}}\rangle\big|_1^2 + \big||\psi_{\mathbf{2}}\rangle\big|_2^2)(\big||\phi_{\mathbf{1}}\rangle\big|_1^2 + \big||\phi_{\mathbf{2}}\rangle\big|_2^2)} \\
&= \frac{2}{\sqrt{2}}\big\||\psi\rangle\big\|\,\big\||\phi\rangle\big\| = \sqrt{2}\big\||\psi\rangle\big\|\,\big\||\phi\rangle\big\|.
\end{aligned}$$

□

Theorem

The constant $\sqrt{2}$ is the best possible in the bicomplex Schwarz inequality.

Proof.

Let us consider $M = \mathbb{M}(2)$. Then $V_1 = \mathbf{e_1}\mathbb{C}(\mathbf{i_1})$ and $V_2 = \mathbf{e_2}\mathbb{C}(\mathbf{i_1})$. Let $|\psi_\mathbf{k}\rangle := \mathbf{e_k} z_{1k}$ and $|\phi_\mathbf{k}\rangle := \mathbf{e_k} z_{2k}$ where $z_{1k}, z_{2k} \in \mathbb{C}(\mathbf{i_1})$ for $k = 1, 2$. Now, consider the following standard scalar product on V_k:

$$\langle\psi_\mathbf{k}|\phi_\mathbf{k}\rangle_{\widehat{k}} := z_{1k}\overline{z_{2k}}$$

for $k = 1, 2$. If we let $|\psi\rangle = |\phi\rangle = \mathbf{e_1}$, then

$$|\langle\psi|\phi\rangle| = \sqrt{2}\,\big\||\psi\rangle\big\|\;\big\||\phi\rangle\big\|$$

since $\langle\psi|\phi\rangle = \langle\mathbf{e_1}|\mathbf{e_1}\rangle_{\widehat{1}}\,\mathbf{e_1} + \langle 0|0\rangle_{\widehat{2}}\,\mathbf{e_2} = \mathbf{e_1}$ and $\big\||\psi\rangle\big\| = \big\||\phi\rangle\big\| = \frac{1}{\sqrt{2}}$. $\square$

- In this section we investigate more specific $\mathbb{M}(2)$-modules, namely those that have a countable basis.

Definition

Let M be a normed $\mathbb{M}(2)$-module. We say that M has a **Schauder $\mathbb{M}(2)$-basis** if there exists a countable set $\{|\psi_1\rangle \dots |\psi_n\rangle \dots\}$ of elements of M such that every element $|\psi\rangle \in M$ admits a unique decomposition as the sum of a convergent series $|\psi\rangle = \sum_{n=1}^{\infty} w_n|\psi_n\rangle$, $w_n \in \mathbb{M}(2)$.

- A normed $\mathbb{M}(2)$-module with a Schauder $\mathbb{M}(2)$-basis is called a **countable $\mathbb{M}(2)$-module**. In this context, it is always possible to construct an orthonormal Schauder $\mathbb{M}(2)$-basis in M.

Theorem (Orthonormalization)

Let M be a bicomplex Hilbert space and let $\{|\psi_n\rangle\}$ be an arbitrary Schauder $\mathbb{M}(2)$-basis of M. Then $\{|\psi_n\rangle\}$ can always be orthonormalized.

- It is interesting to note that the normalizability of kets requires that the scalar product belongs to $\mathbb{D}^+ := \{\alpha \mathbf{e_1} + \beta \mathbf{e_2} | \alpha, \beta > 0\}$. Moreover, this is a necessary condition to recover the standard quantum mechanics from the bicomplex one.

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Proof.

Let us write $\langle\psi_1|\psi_1\rangle = a_{\hat{1}}\mathbf{e_1} + a_{\hat{2}}\mathbf{e_2}$ with $a_{\hat{1}}, a_{\hat{2}} \in \mathbb{R}$, and let

$$|\psi_1'\rangle = (z_{\hat{1}}\mathbf{e_1} + z_{\hat{2}}\mathbf{e_2})|\psi_1\rangle,$$

with $z_{\hat{1}}, z_{\hat{2}} \in \mathbb{C}(\mathbf{i_1})$ and $z_{\hat{1}} \neq 0 \neq z_{\hat{2}}$. We get

$$\begin{aligned}
\langle\psi_1'|\psi_1'\rangle &= (|z_{\hat{1}}|^2\mathbf{e_1} + |z_{\hat{2}}|^2\mathbf{e_2})\,\langle\psi_1|\psi_1\rangle \\
&= (|z_{\hat{1}}|^2\mathbf{e_1} + |z_{\hat{2}}|^2\mathbf{e_2})(a_{\hat{1}}\mathbf{e_1} + a_{\hat{2}}\mathbf{e_2}) \\
&= c_{\hat{1}}a_{\hat{1}}\mathbf{e_1} + c_{\hat{2}}a_{\hat{2}}\mathbf{e_2},
\end{aligned}$$

with $c_{\hat{k}} = |z_{\hat{k}}|^2 \in \mathbb{R}^+$. The normalization condition of $|\psi_1'\rangle$ becomes

$$c_{\hat{1}}a_{\hat{1}}\mathbf{e_1} + c_{\hat{2}}a_{\hat{2}}\mathbf{e_2} = 1,$$

or $c_{\hat{1}}a_{\hat{1}} = 1 = c_{\hat{2}}a_{\hat{2}}$. This is possible only if $a_{\hat{1}} > 0$ and $a_{\hat{2}} > 0$. In other words, $\langle\psi_1|\psi_1\rangle \in \mathbb{D}^+$. □

We conclude this section with this following characterization of the series convergence in M.

Theorem

Let $\{|\psi_n\rangle\}$ be an orthonormal sequence in the bicomplex Hilbert space M and let $\{\alpha_n\}$ be a sequence of bicomplex numbers. Then the series $\sum_{n=1}^{\infty} \alpha_n|\psi_n\rangle$ converges in M if and only if $\sum_{n=1}^{\infty} |\alpha_n|^2$ converges in $\mathbb{R}$.

- In this section, we explain more precisely the relationship between M and M'. First, it is easy to show that V_1 is orthogonal to V_2 in $(M, (\cdot,\cdot))$ and $(M', (\cdot,\cdot)')$ where

Definition

$$\begin{aligned}(|\psi\rangle, |\phi\rangle)' &= \langle\psi|\phi\rangle' \\ &:= \frac{1}{2}\left[\langle\psi_{\mathbf{1}}|\phi_{\mathbf{1}}\rangle_{\widehat{1}} + \langle\psi_{\mathbf{2}}|\phi_{\mathbf{2}}\rangle_{\widehat{2}}\right].\end{aligned}$$

Note: With this definition, M and M' give the same norm.

- In fact, $V_1^{\perp} = V_2$. Therefore, the same symbol $\perp$ can used for M and M', and we have

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Proof.

By definition of the orthogonal complement, we have:

$$V_1^{\perp} = \{|\phi\rangle \in M' |\, \langle\psi|\phi\rangle' = 0 \text{ for all } |\psi\rangle \in V_1\}$$

and

$$V_1^{\perp} = \{|\phi\rangle \in M |\, \langle\psi|\phi\rangle = 0 \text{ for all } |\psi\rangle \in V_1\}.$$

However, $\langle\psi|\phi\rangle' = \frac{1}{2}\left[\langle\psi_{\mathbf{1}}|\phi_{\mathbf{1}}\rangle_{\widehat{1}} + \langle 0|\phi_{\mathbf{2}}\rangle_{\widehat{2}}\right] = 0\ \forall|\psi\rangle \in V_1$ if and only if $\langle\psi_{\mathbf{1}}|\phi_{\mathbf{1}}\rangle_{\widehat{1}} = 0\ \forall|\psi_{\mathbf{1}}\rangle \in V_1$. Therefore, $|\phi_{\mathbf{1}}\rangle = 0$ and $|\phi\rangle \in V_2$. Now,

$$\langle\psi|\phi\rangle = \langle\psi_{\mathbf{1}}|\phi_{\mathbf{1}}\rangle_{\widehat{1}}\,\mathbf{e_1} + \langle 0|\phi_{\mathbf{2}}\rangle_{\widehat{2}}\,\mathbf{e_2} = 0$$

$\forall|\psi\rangle \in V_1$ if and only if $\langle\psi_{\mathbf{1}}|\phi_{\mathbf{1}}\rangle_{\widehat{1}}\,\mathbf{e_1} = 0\ \forall|\psi_{\mathbf{1}}\rangle \in V_1$. Since, $\langle\psi_{\mathbf{1}}|\phi_{\mathbf{1}}\rangle_{\widehat{1}} \in \mathbb{C}(\mathbf{i_1})$, $|\phi\rangle$ must be in V_2. Hence,

$$M = V_1 \oplus V_1^{\perp} = V_1 \oplus V_2 = M'.$$

□

However, this is not the case for the subspace V. Let $\{|\psi_1\rangle \dots |\psi_n\rangle \dots\}$ be a Schauder $\mathbb{M}(2)$-basis associated with the bicomplex Hilbert space $\{M, \langle\cdot|\cdot\rangle\}$. That is, any element $|\psi\rangle$ of M can be written as

$$|\psi\rangle = \sum_{n=1}^{\infty} w_n |\psi_n\rangle, \tag{6}$$

with $w_n \in \mathbb{M}(2)$. As was shown for the finite-dimensional case, an important subset V of M is the set of all kets for which all w_n in (6) belong to $\mathbb{C}(\mathbf{i_1})$. It is obvious that V is a non-empty normed vector space over complex numbers with Schauder basis $\{|\psi_1\rangle \dots |\psi_n\rangle \dots\}$.

From previous result, we see that if $\{|\psi_1\rangle \ldots |\psi_n\rangle \ldots\}$ is an **orthonormal** Schauder $\mathbb{M}(2)$-basis and

$$\sum_{n=1}^{\infty}(\mathbf{e_1} z_{n\widehat{1}} + \mathbf{e_2} z_{n\widehat{2}})|\psi_n\rangle = \sum_{n=1}^{\infty}\mathbf{e_1} z_{n\widehat{1}}|\psi_n\rangle + \sum_{n=1}^{\infty}\mathbf{e_2} z_{n\widehat{2}}|\psi_n\rangle$$

converges in M, then the series

$$\sum_{n=1}^{\infty}|\mathbf{e_1} z_{n\widehat{1}} + \mathbf{e_2} z_{n\widehat{2}}|^2$$

converges in $\mathbb{R}$. In particular, $\sum_{n=1}^{\infty}|z_{n\widehat{k}}|^2$ also converges. Hence $\sum_{n=1}^{\infty} z_{n\widehat{k}}|\psi_n\rangle$ converges and this allows to define projectors P_1 and P_2 from M to V as

$$P_k\left(|\psi\rangle\right) := \sum_{n=1}^{\infty} z_{n\widehat{k}}|\psi_n\rangle, \qquad k = 1, 2.$$

Therefore, any $|\psi\rangle \in M$ can be decomposed uniquely as

$$|\psi\rangle = \mathbf{e_1} P_1\left(|\psi\rangle\right) + \mathbf{e_2} P_2\left(|\psi\rangle\right)$$

and $V_k = \mathbf{e_k} V$ for $k = 1, 2$.

- As in the finite-dimensional case, one can easily show that ket projectors and idempotent-basis projectors (denoted with the same symbol) satisfy the following, for $k = 1, 2$:

Property

$$P_k\left(s|\psi\rangle + t|\phi\rangle\right) = P_k\left(s\right)P_k\left(|\psi\rangle\right) + P_k\left(t\right)P_k\left(|\phi\rangle\right).$$

OPEN QUESTION:

Is it possible to define the projectors P_1 and P_2 from M to V when $\{|\psi_1\rangle \dots |\psi_n\rangle \dots\}$ is NOT **orthonormal**?

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$$P_k\left(s|\psi\rangle + t|\phi\rangle\right) = P_k\left(s\right)P_k\left(|\psi\rangle\right) + P_k\left(t\right)P_k\left(|\phi\rangle\right).$$

OPEN QUESTION:

Is it possible to define the projectors P_1 and P_2 from M to V when $\{|\psi_1\rangle \dots |\psi_n\rangle \dots\}$ is NOT **orthonormal**?

Definition

Let $\{|\psi_n\rangle\}$ be an orthonormal Schauder $\mathbb{M}(2)$-basis of M and let V be the associated vector space. We say that a scalar product is $\mathbb{C}(\mathbf{i_1})$-closed under V if, $\forall|\psi\rangle, |\phi\rangle \in V$, we have $\langle\psi|\phi\rangle \in \mathbb{C}(\mathbf{i_1})$.

We can prove that if the scalar product is $\mathbb{C}(\mathbf{i_1})$-closed under V then the inner space $(V, ||\cdot||)$ is closed in M. Hence, since any closed linear subspace of a Hilbert space satisfy the **Projection Theorem**, we have that

$$M' = V \oplus V^{\perp}$$

when the scalar product is $\mathbb{C}(\mathbf{i_1})$-closed under V.

In this case, it is easy to verify that the orthogonal complement of V for $(M', \langle\cdot|\cdot\rangle')$ is

$$V^{\perp} = \{\mathbf{e_1}|\psi\rangle - \mathbf{e_2}|\psi\rangle : |\psi\rangle \in V\}.$$

Proof.

Let

$$V^{\perp} = \{|\phi\rangle \in M' |\, \langle\psi_{\mathbf{1}}|\phi_{\mathbf{1}}\rangle_{\widehat{1}} + \langle\psi_{\mathbf{2}}|\phi_{\mathbf{2}}\rangle_{\widehat{2}} = 0 \text{ for all } |\psi\rangle \in V\}.$$

By definition of V, we have that

$$|\psi\rangle = P_1\left(|\psi\rangle\right)\mathbf{e_1} + P_1\left(|\psi\rangle\right)\mathbf{e_2}.$$

Therefore, $\langle\psi_{\mathbf{1}}|\phi_{\mathbf{1}}\rangle_{\widehat{1}} + \langle\psi_{\mathbf{2}}|\phi_{\mathbf{2}}\rangle_{\widehat{2}} = 0$ if and only if $\left(P_1\left(|\psi\rangle\right), P_1\left(|\phi\rangle\right)\right)_{\widehat{1}} + \left(P_1\left(|\psi\rangle\right), P_2\left(|\phi\rangle\right)\right)_{\widehat{2}} = 0$. Moreover, since the scalar product is $\mathbb{C}(\mathbf{i_1})$-closed under V then

$$\left(P_1\left(|\psi\rangle\right), P_1\left(|\phi\rangle\right)\right)_{\widehat{1}} = \left(P_1\left(|\psi\rangle\right), P_2\left(|\phi\rangle\right)\right)_{\widehat{2}}$$

for all $|\psi\rangle \in V$. Hence,

$$\left(P_1\left(|\psi\rangle\right), P_1\left(|\phi\rangle\right) + P_2\left(|\phi\rangle\right)\right)_{\widehat{1}} = 0$$

for all $|\psi\rangle \in V$. Then, $P_1\left(|\phi\rangle\right) = -P_2\left(|\phi\rangle\right)$ and $|\psi\rangle = \mathbf{e_1}|\psi\rangle - \mathbf{e_2}|\psi\rangle$. □

This is not the case for $(M, \langle\cdot|\cdot\rangle)$ since the orthogonal complement of V is $\{0\}$. In fact, since M is not a Hilbert space, the Projection Theorem cannot be applied.

Proof.

Let

$$V^{\perp} = \{|\phi\rangle \in M|\ \langle\psi_{\mathbf{1}}|\phi_{\mathbf{1}}\rangle_{\widehat{1}}\,\mathbf{e_1} + \langle\psi_{\mathbf{2}}|\phi_{\mathbf{2}}\rangle_{\widehat{2}}\,\mathbf{e_2} = 0 \text{ for all } |\psi\rangle \in V\}.$$

By definition, the scalar products are in $\mathbb{C}(\mathbf{i_1})$, then we have that

$$\langle\psi_{\mathbf{1}}|\phi_{\mathbf{1}}\rangle_{\widehat{1}}\,\mathbf{e_1} + \langle\psi_{\mathbf{2}}|\phi_{\mathbf{2}}\rangle_{\widehat{2}}\,\mathbf{e_2} = 0$$

$\forall|\psi\rangle \in V$ if and only if $\langle\psi_{\mathbf{1}}|\phi_{\mathbf{1}}\rangle_{\widehat{1}} = \langle\psi_{\mathbf{2}}|\phi_{\mathbf{2}}\rangle_{\widehat{2}} = 0$ $\forall|\psi_{\mathbf{1}}\rangle \in V_1$ and $\forall|\psi_{\mathbf{2}}\rangle \in V_2$. Hence, $|\phi_{\mathbf{1}}\rangle = |\phi_{\mathbf{2}}\rangle = 0$ and $|\phi\rangle = 0$. □

- Finally, if we define

Definition

$$V_1^{\dagger_2} := \{\mathbf{e_1} P_2(|\psi\rangle) + \mathbf{e_2} P_1(|\psi\rangle) : |\psi\rangle \in V_1\} = \{\mathbf{e_2} P_1(|\psi\rangle) : |\psi\rangle \in V_1\}$$

where $\dagger_2$ is used as the natural extension of the conjugate $\dagger_2$ in $\mathbb{M}(2)$, we obtain that $V_1^{\dagger_2} = \mathbf{e_2} V = V_2 = V_1^{\perp}$ and

$$M = V_1 \oplus V_1^{\dagger_2} = V_1 \oplus V_2 = M'.$$

- This definition of $\dagger_2$ is **universal** for any element inside a bicomplex Hilbert space with an orthonormal Schauder $\mathbb{M}(2)$-basis, and satisfy the following properties:
  1. $(|\phi\rangle^{\dagger_2})^{\dagger_2} = |\phi\rangle$;
  2. $(|\phi\rangle \pm |\psi\rangle)^{\dagger_2} = |\phi\rangle^{\dagger_2} \pm |\psi\rangle^{\dagger_2}$;
  3. $(w|\phi\rangle)^{\dagger_2} = w^{\dagger_2}|\phi\rangle^{\dagger_2}$

  $\forall |\phi\rangle, |\psi\rangle \in M$ and $\forall w \in \mathbb{M}(2)$.

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- Complex Hilbert spaces are fundamental tools of quantum mechanics. We should therefore expect that bicomplex Hilbert spaces should be relevant to any attempted generalization of quantum mechanics to bicomplex numbers. Let us examine the example of the quantum harmonic oscillator.
- We start with the following function space. Let n be a nonnegative integer and let α be a positive real number. Consider the following function of a real variable x:

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- Let S be the set of all finite linear combinations of functions $f_{n,\alpha}(x)$, with complex coefficients. Furthermore, let a bicomplex function $u(x)$ be defined as

$$u(x) = \mathbf{e_1} u_{\widehat{1}}(x) + \mathbf{e_2} u_{\widehat{2}}(x),$$

where $u_{\widehat{1}}$ and $u_{\widehat{2}}$ are both in S. The set of all functions $u(x)$ is an $\mathbb{M}(2)$-module, denoted by M_S.

- Let $u(x)$ and $v(x)$ both belong to M_S. We define a mapping (u, v) of this pair of functions into $\mathbb{D}_+$ as follows:

$$(u, v) := \int_{-\infty}^{\infty} u^{\dagger_3}(x)v(x)\mathrm{d}x = \int_{-\infty}^{\infty} \left[\mathbf{e_1}\bar{u}_{\widehat{1}}(x)v_{\widehat{1}}(x) + \mathbf{e_2}\bar{u}_{\widehat{2}}(x)v_{\widehat{2}}(x)\right] \mathrm{d}x.$$

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- It is not hard to see that the last equation is always finite and satisfies all the properties of a bicomplex scalar product.
- Let $\xi = \mathbf{e_1}\xi_{\widehat{1}} + \mathbf{e_2}\xi_{\widehat{2}}$ be in $\mathbb{D}^+$ and let us define two operators X *(position)* and P *(momentum)* that act on elements of M_S as follows:

$$X\{u(x)\} := xu(x), \qquad P\{u(x)\} := -\mathbf{i_1}\hbar\xi\frac{\mathrm{d}u(x)}{\mathrm{d}x}.$$

In standard quantum mechanics, the position operator is the operator that corresponds to the position observable of a particle. The eigenvalue of the operator is the position vector of the particle.
- It is not difficult to show the following commutator relation:

$$[X, P] = \mathbf{i_1}\hbar\xi\mathrm{I}.$$

- Note that the action of X and P on elements of M_S always yields elements of M_S. That is, X and P are defined on all of M_S.

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- Let m and ω be two positive real numbers. We define the bicomplex harmonic oscillator Hamiltonian as follows:

$$H := \frac{1}{2m}P^2 + \frac{1}{2}m\omega^2 X^2.$$

- The problem of the bicomplex quantum harmonic oscillator consists in finding the eigenvalues and eigenfunctions of H.
- That problem was solved in a previous paper on the topic. The results can be summarized as follows. Let θ_k $(k = \widehat{1}, \widehat{2})$ be defined as

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- Bicomplex harmonic oscillator eigenfunctions can then be written as (the most general eigenfunction would have different l indices in the two terms l and l'):

$$\phi_l(x) = \mathbf{e_1}\phi_{l\hat{1}} + \mathbf{e_2}\phi_{l\hat{2}}$$
$$= \mathbf{e_1}\left[\sqrt{\frac{m\omega}{\pi\hbar\xi_{\hat{1}}}}\frac{1}{2^l l!}\right]^{1/2} e^{-\theta_{\hat{1}}^2/2} H_l(\theta_{\hat{1}}) + \mathbf{e_1}\left[\sqrt{\frac{m\omega}{\pi\hbar\xi_{\hat{2}}}}\frac{1}{2^l l!}\right]^{1/2} e^{-\theta_{\hat{2}}^2/2} H_l(\theta_{\hat{2}}),$$

where H_l are Hermite polynomials. The last equation can be written more succinctly as

$$\phi_l(x) = \left[\sqrt{\frac{m\omega}{\pi\hbar\xi}}\frac{1}{2^l l!}\right]^{1/2} e^{-\theta^2/2} H_l(\theta),$$

where

$$\theta := \mathbf{e_1}\theta_{\hat{1}} + \mathbf{e_2}\theta_{\hat{2}} \quad \text{and} \quad H_l(\theta) := \mathbf{e_1}H_l(\theta_{\hat{1}}) + \mathbf{e_2}H_l(\theta_{\hat{2}}).$$

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- Another way to express our eigenfunctions in term of real and hyperbolic part is to rewrite the hyperbolic exponential $\mathrm{e}^{-\theta^2/2}$ in term of **real hyperbolic sinus and cosinus**. Indeed, we can write

$$\begin{aligned}\mathrm{e}^{-\theta^2/2} &= \mathrm{e}^{-\frac{(\theta_1^2+\theta_2^2)}{2}}\mathrm{e}^{-\theta_1\theta_2\mathbf{j}}\\ &= \mathrm{e}^{-\frac{(\theta_1^2+\theta_2^2)}{2}}\{\cosh\theta_1\theta_2 - \mathbf{j}\sinh\theta_1\theta_2\} \quad \text{with} \quad \theta = \theta_1+\theta_2\mathbf{j}.\end{aligned}$$

- Taking

$$\xi = \alpha + \beta\mathbf{j},$$

we have that

$$\begin{aligned}\xi^{-1/4} &= \frac{(\alpha+\beta)^{-1/4}+(\alpha-\beta)^{1/4}}{2} + \mathbf{j}\frac{(\alpha+\beta)^{-1/4}-(\alpha-\beta)^{1/4}}{2}\\ &:= \alpha' + \beta'\mathbf{j}.\end{aligned}$$

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$$e^{-\theta^2/2} = e^{-\frac{(\theta_1^2+\theta_2^2)}{2}} e^{-\theta_1\theta_2\mathbf{j}}$$
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we have that

$$\xi^{-1/4} = \frac{(\alpha+\beta)^{-1/4} + (\alpha-\beta)^{1/4}}{2} + \mathbf{j}\frac{(\alpha+\beta)^{-1/4} - (\alpha-\beta)^{1/4}}{2}$$
$$:= \alpha' + \beta'\mathbf{j}.$$

For the normalized eigenfunction, we can then write

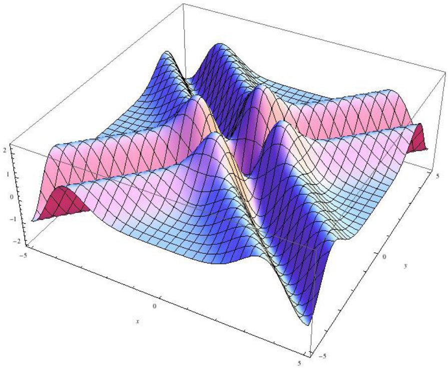
$$\phi_l(x) = \left[\sqrt{\frac{m\omega}{\pi\hbar}}\frac{1}{2^l l!}\right]^{1/2} \mathrm{e}^{-\frac{(\theta_1^2+\theta_2^2)}{2}}$$
$$\cdot\left\{\left[\left(\alpha' \cosh\theta_1\theta_2 - \beta' \sinh\theta_1\theta_2\right)\mathbf{Re}\left(H_l(\theta)\right) + \left(\beta' \cosh\theta_1\theta_2 - \alpha' \sinh\theta_1\theta_2\right)\mathbf{Hy}\left(H_l(\theta)\right)\right]\right.$$
$$\left. + \mathbf{j}\left[\left(\alpha' \cosh\theta_1\theta_2 - \beta' \sinh\theta_1\theta_2\right)\mathbf{Hy}\left(H_l(\theta)\right) + \left(\beta' \cosh\theta_1\theta_2 - \alpha' \sinh\theta_1\theta_2\right)\mathbf{Re}\left(H_l(\theta)\right)\right]\right\},$$

where $\mathbf{Re}\left(H_l(\theta)\right)$ and $\mathbf{Hy}\left(H_l(\theta)\right)$ stand for the real and the hyperbolic part of $H_l(\theta)$, respectively.

In fact, $\mathbf{Re}\left(H_l(\theta)\right) = \mathbf{Re}\left(H_l(x,y)\right)$ and $\mathbf{Hy}\left(H_l(\theta)\right) = \mathbf{Hy}\left(H_l(x,y)\right)$ are **polynomials of two real variables**. For examples:

$$
\begin{aligned}
&\mathbf{Re}\left(H_0(x,y)\right) = 1, && \mathbf{Hy}\left(H_0(x,y)\right) = 0\\
&\mathbf{Re}\left(H_1(x,y)\right) = 2x, && \mathbf{Hy}\left(H_1(x,y)\right) = 2y\\
&\mathbf{Re}\left(H_2(x,y)\right) = 4x^2 + 4y^2 - 2, && \mathbf{Hy}\left(H_2(x,y)\right) = 8xy\\
&\mathbf{Re}\left(H_3(x,y)\right) = 8x^3 + 24xy^2 - 12x, && \mathbf{Hy}\left(H_3(x,y)\right) = 24x^2y + 8y^3 - 12y.
\end{aligned}
$$

Figure 1: $\mathbf{Re}(\phi_2(\theta_1, \theta_2))$ for $\alpha = 1$ and $\beta = 0$



It is not so hard to see that if we take $\xi_{\hat{1}} = 1 = \xi_{\hat{2}}$ (resp. $\alpha = 1$ and $\beta = 0$) and $I = I'$ (indirectly $X_{\hat{1}} = X_{\hat{2}}$, $P_{\hat{1}} = P_{\hat{2}}$ and so on), we recover the usual eigenfunctions and energy of the standard quantum harmonic oscillator for the real slice ($x \in \mathbb{R}$ or $\theta_2 = 0$).

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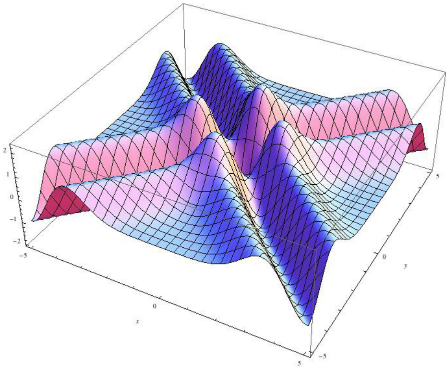

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Figure 2: **Hy** $(\phi_2(\theta_1, \theta_2))$ for $\alpha = 1$ and $\beta = 0$

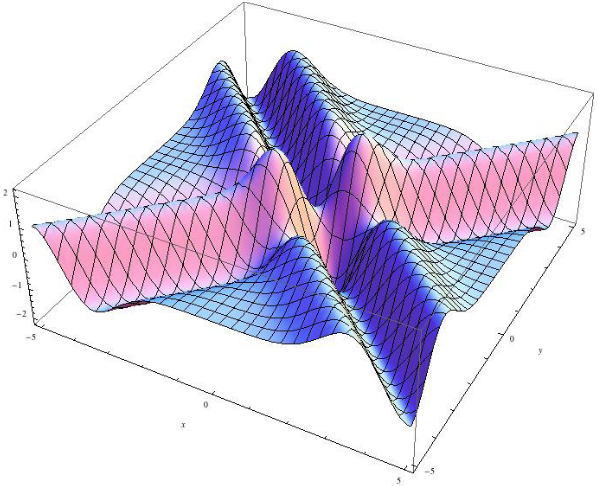
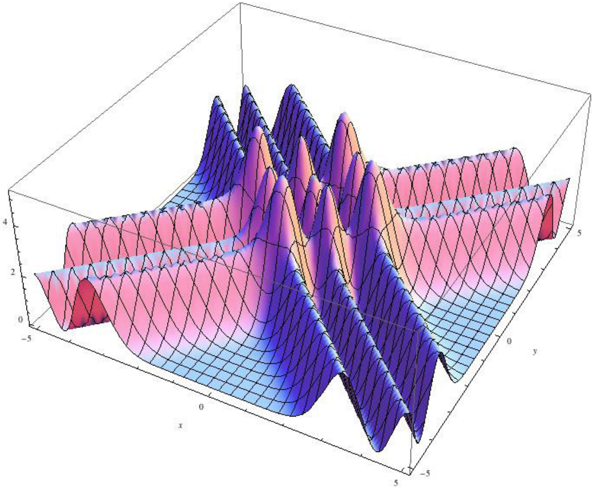


Figure 3: $|\phi_2(\theta_1,\theta_2)|^2 = \mathbf{Re}\left(\phi_2(\theta_1,\theta_2)\right)^2 + \mathbf{Hy}\left(\phi_2(\theta_1,\theta_2)\right)^2$

Here is the **probability density** of $\phi_2(\theta_1,\theta_2)$ for $\alpha = 1$ and $\beta = 0$.



- Finally, we can show that the collection of all finite linear combinations of bicomplex functions $\phi_l(x)$, with bicomplex coefficients, is an $\mathbb{M}(2)$-module. Specifically,

$$\tilde{M} := \left\{ \sum_l w_l \phi_l(x) \mid w_l \in \mathbb{M}(2) \right\}.$$

- Since $\tilde{M}$ only involves finite linear combinations of the functions ϕ_l, it is not complete. With new methods developed recently, however, we can extend $\tilde{M}$ to a complete module, in fact to a bicomplex Hilbert space.

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- We can define two vector spaces $\tilde{V}_1$ and $\tilde{V}_2$ as $\tilde{V}_1 = \mathbf{e_1}\tilde{M}$ and $\tilde{V}_2 = \mathbf{e_2}\tilde{M}$. It is clear that $\tilde{V}_1$ contains all functions $\mathbf{e_1}\phi_{l\hat{1}}$ and $\tilde{V}_2$ contains all $\mathbf{e_2}\phi_{l\hat{2}}$. Now the functions $\phi_{l\hat{1}}$ and $\phi_{l\hat{2}}$ are normalized eigenfunctions of the usual quantum harmonic oscillator (with $\hbar$ replaced by $\hbar\xi_{\hat{1}}$ or $\hbar\xi_{\hat{2}}$). It is well-known that, as a Schauder basis, these eigenfunctions generate $L^2(\mathbb{R})$.
- Let $u(x)$ be defined as before, except that $u_{\hat{1}}(x)$ and $u_{\hat{2}}(x)$ are both taken as $L^2(\mathbb{R})$ functions. Clearly, the set of all $u(x)$ makes up an $\mathbb{M}(2)$-module, which we shall denote by M. With the scalar product, M becomes a bicomplex pre-Hilbert space. Since $L^2(\mathbb{R})$ is complete we obtain:

Corollary

M is a bicomplex Hilbert space.

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Corollary

M is a bicomplex Hilbert space.

- We can define two vector spaces $\tilde{V}_1$ and $\tilde{V}_2$ as $\tilde{V}_1 = \mathbf{e_1}\tilde{M}$ and $\tilde{V}_2 = \mathbf{e_2}\tilde{M}$. It is clear that $\tilde{V}_1$ contains all functions $\mathbf{e_1}\phi_{l\hat{1}}$ and $\tilde{V}_2$ contains all $\mathbf{e_2}\phi_{l\hat{2}}$. Now the functions $\phi_{l\hat{1}}$ and $\phi_{l\hat{2}}$ are normalized eigenfunctions of the usual quantum harmonic oscillator (with $\hbar$ replaced by $\hbar\xi_{\hat{1}}$ or $\hbar\xi_{\hat{2}}$). It is well-known that, as a Schauder basis, these eigenfunctions generate $L^2(\mathbb{R})$.
- Let $u(x)$ be defined as before, except that $u_{\hat{1}}(x)$ and $u_{\hat{2}}(x)$ are both taken as $L^2(\mathbb{R})$ functions. Clearly, the set of all $u(x)$ makes up an $\mathbb{M}(2)$-module, which we shall denote by M. With the scalar product, M becomes a bicomplex pre-Hilbert space. Since $L^2(\mathbb{R})$ is complete we obtain:

Corollary

M is a bicomplex Hilbert space.

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