

Hilbert Space of the Bicomplex Quantum Harmonic Oscillator

Raphaël Gervais Lavoie*, Louis Marchildon* and Dominic Rochon†

**Département de physique, Université du Québec, Trois-Rivières (Québec), Canada G9A 5H7*

†*Département de mathématiques et d'informatique, Université du Québec, Trois-Rivières (Québec), Canada G9A 5H7*

Abstract. Bicomplex numbers are pairs of complex numbers with a multiplication law that makes them a commutative ring. The problem of the quantum harmonic oscillator is investigated in the framework of bicomplex numbers. Starting with the commutator of the bicomplex position and momentum operators, we find eigenvalues and eigenkets of the bicomplex harmonic oscillator Hamiltonian. Coordinate-basis eigenfunctions of the Hamiltonian are then obtained in terms of hyperbolic Hermite polynomials, and some of them are graphically illustrated. These eigenfunctions form a basis of an infinite-dimensional module over bicomplex numbers, and this module can be given the structure of a bicomplex Hilbert space.

Keywords: Bicomplex numbers, bicomplex quantum mechanics, modules, Hilbert spaces, harmonic oscillator

PACS: 02.10.Hh, 03.65.-w, 03.65.Fd

INTRODUCTION

The mathematical structure of quantum mechanics consists in Hilbert spaces defined over the field of complex numbers [1]. This structure has been extremely successful in explaining a bewildering amount of experimental data. Yet it seems that there is no compelling argument restricting the number system on which quantum mechanics is built to \mathbb{C} . It has been pointed out long ago [2] that quantum mechanics can be extended to quaternions (\mathbb{H}), a proposal that has since been developed substantially [3].

Complex numbers and quaternions are both division algebras, which means that neither on \mathbb{C} nor on \mathbb{H} can one find products of nonzero numbers that vanish. Moreover, on both \mathbb{C} and \mathbb{H} one can define a norm so that $N(w w') = N(w)N(w')$. This property is particularly useful to represent quantum-mechanical probabilities and the correspondence principle with classical mechanics. Yet one can ask whether it is possible to extend quantum mechanics, or at least part of it, to number systems that don't obey it. Several possibilities have been considered but we will focus here on the system of bicomplex numbers [4].

Bicomplex quantum mechanics was first investigated in [5, 6]. Motivated by these results, we examined the specific problem of the bicomplex quantum harmonic oscillator and were able to obtain eigenvalues and eigenfunctions in full generality [7]. This was then put in proper mathematical context through an investigation of finite-dimensional [8] and infinite-dimensional [9] bicomplex Hilbert spaces and modules. The purpose of this paper is to present an overview of these results, with special emphasis on the harmonic oscillator eigenfunctions and the kind of Hilbert space they generate.

BICOMPLEX NUMBERS AND MODULES

This section summarizes basic properties of bicomplex numbers and finite-dimensional modules defined over them. Proofs and additional material can be found in [4, 6, 8, 10].

Quaternions and bicomplex numbers

Bicomplex numbers (\mathbb{T}) can be introduced through comparison with the better known quaternions. A *quaternion* q is defined in terms of four real numbers $q_0, q_1, q_2,$ and q_3 as

$$q = q_0 + q_1 e_1 + q_2 e_2 + q_3 e_3, \quad (1)$$

where $e_1, e_2,$ and e_3 are three units satisfying the following product rule:

$$e_i e_j = -\delta_{ij} + \sum_k \varepsilon_{ijk} e_k. \quad (2)$$

Here δ_{ij} is the Kronecker delta and ε_{ijk} is the totally antisymmetric Levi-Civita symbol. The set of all quaternions is a noncommutative field and it makes up a division algebra.

A *bicomplex number* w is also defined in terms of four real numbers $w_0, w_1, w_2,$ and w_3 as

$$w = w_0 + w_1 \mathbf{i}_1 + w_2 \mathbf{i}_2 + w_3 \mathbf{j}. \quad (3)$$

Here $\mathbf{i}_1, \mathbf{i}_2,$ and \mathbf{j} are three commutative units satisfying the following product rule:

$$(\mathbf{i}_1)^2 = -1 = (\mathbf{i}_2)^2, \quad \mathbf{j}^2 = 1, \quad \mathbf{i}_1 \mathbf{i}_2 = \mathbf{j}, \quad \mathbf{i}_1 \mathbf{j} = -\mathbf{i}_2, \quad \mathbf{i}_2 \mathbf{j} = -\mathbf{i}_1. \quad (4)$$

By contrast with \mathbb{H} , the set \mathbb{T} in a commutative ring that does not make up a division algebra.

Bicomplex algebra

Two subalgebras of \mathbb{T} are isomorphic with the complex numbers. The first one, denoted by $\mathbb{C}(\mathbf{i}_1)$, contains all w such that $w = w_0 + w_1 \mathbf{i}_1$. The second one, $\mathbb{C}(\mathbf{i}_2)$, contains all $w = w_0 + w_2 \mathbf{i}_2$. A third subalgebra of \mathbb{T} , denoted by \mathbb{D} , contains all w such that $w = w_0 + w_3 \mathbf{j}$. These are the so-called *hyperbolic numbers*. The set \mathbb{D} has an important subset \mathbb{D}^+ , where w_0 and w_3 are both non-negative.

Bicomplex algebra is considerably simplified by the introduction of two bicomplex numbers \mathbf{e}_1 and \mathbf{e}_2 defined as

$$\mathbf{e}_1 = \frac{1 + \mathbf{j}}{2}, \quad \mathbf{e}_2 = \frac{1 - \mathbf{j}}{2}. \quad (5)$$

One easily checks that

$$(\mathbf{e}_1)^2 = \mathbf{e}_1, \quad (\mathbf{e}_2)^2 = \mathbf{e}_2, \quad \mathbf{e}_1 + \mathbf{e}_2 = 1, \quad \mathbf{e}_1 \mathbf{e}_2 = 0. \quad (6)$$

The last of (6) illustrates the fact that \mathbb{T} is not a division algebra.

Any $w \in \mathbb{T}$ can be written uniquely as $w = z_1 \mathbf{e}_1 + z_2 \mathbf{e}_2$ with $z_1, z_2 \in \mathbb{C}(\mathbf{i}_1)$. The complex numbers z_1 and z_2 are called the *idempotent projections* of w . A bicomplex number w such that either $z_1 = 0$ or $z_2 = 0$ is said to be in the *null cone*, and such a w has no multiplicative inverse. One can show that (for any polynomial Q)

$$Q(w) = \mathbf{e}_1 Q(z_1) + \mathbf{e}_2 Q(z_2), \quad (7)$$

$$\exp(w) = \mathbf{e}_1 \exp(z_1) + \mathbf{e}_2 \exp(z_2), \quad (8)$$

$$\sqrt[n]{w} = \mathbf{e}_1 \sqrt[n]{z_1} + \mathbf{e}_2 \sqrt[n]{z_2}. \quad (9)$$

The algebra of bicomplex numbers is in fact isomorphic with the algebra of complex two-dimensional diagonal matrices.

The *conjugate* of a bicomplex number can be defined in several ways [10]. The definition most useful for our purposes is the following:

$$w^\dagger = \bar{z}_1 \mathbf{e}_1 + \bar{z}_2 \mathbf{e}_2. \quad (10)$$

Here \bar{z}_1 and \bar{z}_2 are the usual complex conjugates of z_1 and z_2 . Note that for any w , the product $ww^\dagger = z_1 \bar{z}_1 \mathbf{e}_1 + z_2 \bar{z}_2 \mathbf{e}_2$ belongs to \mathbb{D}^+ . We define the *modulus* of w as

$$|w| = \frac{1}{\sqrt{2}} \sqrt{z_1 \bar{z}_1 + z_2 \bar{z}_2}. \quad (11)$$

Clearly, $|w| \geq 0$ and $|w| = 0$ if and only if $w = 0$. One can show that

$$|w + w'| \leq |w| + |w'|, \quad |ww'| \leq \sqrt{2} |w| |w'|, \quad (12)$$

but in general $|ww'| \neq |w| |w'|$. Nor is $|ww^\dagger|$ in general equal to $|w|^2$, which means that the bicomplex algebra \mathbb{T} is not a C^* -algebra.

Modules, scalar product and linear operators

A module is just like a vector space, but it is defined over a ring instead of a field. A \mathbb{T} -*module* is defined as a module over \mathbb{T} . Let M be a \mathbb{T} -module. We will use the ket notation $|\psi\rangle$ to denote a generic element of M . Note that $|\psi\rangle$ is not a vector, since vector spaces are defined over fields. However, one can show that any $|\psi\rangle$ can be decomposed uniquely as

$$|\psi\rangle = \mathbf{e}_1 |\psi\rangle + \mathbf{e}_2 |\psi\rangle \equiv |\psi\rangle_1 + |\psi\rangle_2. \quad (13)$$

Henceforth a bold index will always denote a quantity that results from multiplication by \mathbf{e}_1 or \mathbf{e}_2 , and therefore satisfies $\mathbf{e}_k |\psi\rangle_k = |\psi\rangle_k$ ($\mathbf{k} = \mathbf{1}, \mathbf{2}$). The set of all $|\psi\rangle_1$ makes up a genuine vector space over $\mathbb{C}(\mathbf{i}_1)$, and similarly with the set of all $|\psi\rangle_2$.

A *bicomplex scalar* (or *inner*) *product* maps two arbitrary kets $|\psi\rangle$ and $|\phi\rangle$ into a bicomplex number $(|\psi\rangle, |\phi\rangle)$, so that the following always holds (for any s in \mathbb{T}):

1. $(|\psi\rangle, |\phi\rangle + |\chi\rangle) = (|\psi\rangle, |\phi\rangle) + (|\psi\rangle, |\chi\rangle)$;

2. $(|\psi\rangle, s|\phi\rangle) = s(|\psi\rangle, |\phi\rangle)$;
3. $(|\psi\rangle, |\phi\rangle) = (|\phi\rangle, |\psi\rangle)^\dagger$;
4. $(|\psi\rangle, |\psi\rangle) \in \mathbb{D}^+$ and $(|\psi\rangle, |\psi\rangle) = 0$ iff $|\psi\rangle = 0$.

A *linear operator* is defined so that (for any s and t in \mathbb{T})

$$A(s|\psi\rangle + t|\phi\rangle) = sA|\psi\rangle + tA|\phi\rangle. \quad (14)$$

Linear operators can be decomposed uniquely as $A = \mathbf{e}_1 A + \mathbf{e}_2 A = A_1 + A_2$, and one can show that

$$A|\psi\rangle = A_1|\psi\rangle_1 + A_2|\psi\rangle_2. \quad (15)$$

The *adjoint* A^* of a linear operator A is the unique operator such that for any kets $|\psi\rangle$ and $|\phi\rangle$,

$$(|\psi\rangle, A|\phi\rangle) = (A^*|\psi\rangle, |\phi\rangle). \quad (16)$$

In finite-dimensional modules, the adjoint is always defined. A *self-adjoint* operator is an operator that coincides with its adjoint. One can show [6, 8] that in finite-dimensional modules, eigenvalues of a self-adjoint operator are hyperbolic numbers, and eigenkets are orthogonal if associated eigenvalues differ in both idempotent projections.

BICOMPLEX HARMONIC OSCILLATOR

The Hamiltonian of the standard quantum harmonic oscillator is defined as [11]

$$H = \frac{1}{2m}P^2 + \frac{1}{2}m\omega^2X^2, \quad (17)$$

where m and ω are positive real numbers and X and P are self-adjoint operators satisfying a canonical commutation relation.

The most straightforward generalization to bicomplex numbers consists in taking X and P as self-adjoint operators acting in a \mathbb{T} -module, and satisfying the commutation relation

$$[X, P] = \mathbf{i}_1 \hbar \xi I, \quad (18)$$

with $\xi \in \mathbb{T}$. The self-adjointness of X and P requires that $\xi \in \mathbb{D}$, that is, $\xi = \xi_1 \mathbf{e}_1 + \xi_2 \mathbf{e}_2$, with ξ_1 and ξ_2 real. One can show [7] that rescaling X and P and redefining m and ω can make both ξ_1 and ξ_2 positive and can make one of them (but not both) equal to 1. Without loss of generality then, we let $\xi_1 > 0$ and $\xi_2 > 0$.

We now proceed to obtain eigenvalues and eigenkets of the bicomplex Hamiltonian H , following the constructive method used in [7].

Creation and annihilation operators

Just as in the standard case, we define an operator A and its adjoint A^* as

$$A = \frac{1}{\sqrt{2m\hbar\omega}}(m\omega X + \mathbf{i}_1 P), \quad A^* = \frac{1}{\sqrt{2m\hbar\omega}}(m\omega X - \mathbf{i}_1 P). \quad (19)$$

In terms of A and A^* the Hamiltonian becomes

$$H = \hbar\omega \left(A^*A + \frac{\xi}{2}I \right) = \hbar\omega \left(AA^* - \frac{\xi}{2}I \right). \quad (20)$$

The following commutation relations can easily be verified:

$$[A, A^*] = \xi I, \quad [H, A] = -\hbar\omega \xi A, \quad [H, A^*] = \hbar\omega \xi A^*. \quad (21)$$

Eigenvalue equation

The eigenvalue equation for the Hamiltonian can be written as $H|E\rangle = E|E\rangle$. To solve it, we will use the idempotent decomposition of bicomplex numbers, kets, and operators introduced before. We write

$$H = \mathbf{e}_1 H + \mathbf{e}_2 H = H_1 + H_2, \quad (22)$$

$$E = \mathbf{e}_1 E + \mathbf{e}_2 E = E_1 + E_2, \quad (23)$$

$$|E\rangle = \mathbf{e}_1 |E\rangle + \mathbf{e}_2 |E\rangle = |E\rangle_1 + |E\rangle_2. \quad (24)$$

One can easily show that

$$H_1 |E\rangle_1 = E_1 |E\rangle_1, \quad H_2 |E\rangle_2 = E_2 |E\rangle_2. \quad (25)$$

Moreover, all previous relations such as the definition of H , A , and A^* as well as the commutation relations project on $\mathbf{1}$ and $\mathbf{2}$.

By an argument similar to the one used in the case of the standard harmonic oscillator, one can show that

$$H_{\mathbf{k}} A_{\mathbf{k}}^l |E\rangle_{\mathbf{k}} = (E_{\mathbf{k}} - l\hbar\omega \xi_{\mathbf{k}}) A_{\mathbf{k}}^l |E\rangle_{\mathbf{k}}, \quad (26)$$

$$H_{\mathbf{k}} (A_{\mathbf{k}}^*)^l |E\rangle_{\mathbf{k}} = (E_{\mathbf{k}} + l\hbar\omega \xi_{\mathbf{k}}) (A_{\mathbf{k}}^*)^l |E\rangle_{\mathbf{k}}. \quad (27)$$

From a given projected eigenket, this generates new projected eigenkets with lower or higher projected eigenvalues. This justifies calling A an *annihilation* and A^* a *creation* operator. One can also show that

$$\left(A_{\mathbf{k}}^{l+1} |E\rangle_{\mathbf{k}}, A_{\mathbf{k}}^{l+1} |E\rangle_{\mathbf{k}} \right) = \left\{ \frac{E_{\mathbf{k}}}{\hbar\omega} - \left(l + \frac{1}{2} \right) \xi_{\mathbf{k}} \right\} \left(A_{\mathbf{k}}^l |E\rangle_{\mathbf{k}}, A_{\mathbf{k}}^l |E\rangle_{\mathbf{k}} \right). \quad (28)$$

Equation (28) implies that the lowering of eigenvalues cannot go on forever, for otherwise the scalar product will no longer be in \mathbb{D}^+ .

Hence there must exist a ket $|\phi_0\rangle_{\mathbf{k}}$ such that

$$A_{\mathbf{k}} |\phi_0\rangle_{\mathbf{k}} = 0. \quad (29)$$

Making use of the projection of (20), one easily shows that

$$H_{\mathbf{k}} |\phi_0\rangle_{\mathbf{k}} = \frac{1}{2} \hbar\omega \xi_{\mathbf{k}} |\phi_0\rangle_{\mathbf{k}}. \quad (30)$$

Let us now define

$$|\phi_l\rangle_1 = \frac{1}{\sqrt{l!\xi_1^l}}(A_1^*)^l|\phi_0\rangle_1, \quad |\phi_l\rangle_2 = \frac{1}{\sqrt{l!\xi_2^l}}(A_2^*)^l|\phi_0\rangle_2. \quad (31)$$

Letting

$$|\phi_l\rangle = |\phi_l\rangle_1 + |\phi_l\rangle_2, \quad (32)$$

one finds that

$$H|\phi_l\rangle = (l+1/2)\hbar\omega\xi|\phi_l\rangle. \quad (33)$$

Defining a ket $|\psi\rangle$ as *normalized* if $(|\psi\rangle, |\psi\rangle) = 1$, one can show [7] that $|\phi_l\rangle$ is normalized if $|\phi_0\rangle$ is. Operators A and A^* act on $|\phi_l\rangle$ as

$$A^*|\phi_l\rangle = \sqrt{(l+1)\xi}|\phi_{l+1}\rangle, \quad A|\phi_l\rangle = \sqrt{l\xi}|\phi_{l-1}\rangle. \quad (34)$$

The most general eigenket of H is given by

$$|\phi\rangle = w_{l1}|\phi_l\rangle_1 + w_{l'2}|\phi_{l'}\rangle_2, \quad (35)$$

where w_{l1} and $w_{l'2}$ are arbitrary complex numbers. This eigenket corresponds to the eigenvalue

$$\lambda = \hbar\omega \left\{ \left(l + \frac{1}{2} \right) \xi_1 \mathbf{e}_1 + \left(l' + \frac{1}{2} \right) \xi_2 \mathbf{e}_2 \right\}. \quad (36)$$

For general eigenkets of H to be orthogonal, it is not enough that eigenvalues be different. Both l and l' must be different.

It was shown in [7] that the set of all finite linear combinations of kets $|\phi_l\rangle$, with bicomplex coefficients, is an infinite-dimensional free \mathbb{T} -module.

Eigenfunctions

A bicomplex function of a real variable can be defined as

$$u(x) = \mathbf{e}_1 u_1(x) + \mathbf{e}_2 u_2(x), \quad (37)$$

where u_1 and u_2 are $\mathbb{C}(\mathbf{i}_1)$ -valued functions defined in an appropriate function space. A scalar product can be specified as

$$\begin{aligned} (u, v) &= \int_{-\infty}^{\infty} u^\dagger(x)v(x)dx \\ &= \int_{-\infty}^{\infty} [\mathbf{e}_1 \bar{u}_1(x)v_1(x) + \mathbf{e}_2 \bar{u}_2(x)v_2(x)] dx. \end{aligned} \quad (38)$$

Operators X and P can be realized on the function space as

$$X\{u(x)\} = xu(x), \quad P\{u(x)\} = -\mathbf{i}_1 \hbar \xi \frac{du(x)}{dx}. \quad (39)$$

The Hamiltonian (17) then becomes a differential operator acting in the function space.

One can show [7] that the normalized eigenfunction of H corresponding to the eigenket (32) is given by

$$\phi_l(x) = \mathbf{e}_1 \left[\sqrt{\frac{m\omega}{\pi\hbar\xi_1}} \frac{1}{2^l l!} \right]^{1/2} e^{-\theta_1^2/2} H_l(\theta_1) + \mathbf{e}_2 \left[\sqrt{\frac{m\omega}{\pi\hbar\xi_2}} \frac{1}{2^l l!} \right]^{1/2} e^{-\theta_2^2/2} H_l(\theta_2), \quad (40)$$

where the H_l are Hermite polynomials and

$$\theta_1 = \sqrt{\frac{m\omega}{\hbar\xi_1}} x, \quad \theta_2 = \sqrt{\frac{m\omega}{\hbar\xi_2}} x. \quad (41)$$

Letting

$$\theta = \mathbf{e}_1 \theta_1 + \mathbf{e}_2 \theta_2, \quad H_l(\theta) = \mathbf{e}_1 H_l(\theta_1) + \mathbf{e}_2 H_l(\theta_2), \quad (42)$$

and making use of (7)–(9), one finds that

$$\phi_l(x) = \left[\sqrt{\frac{m\omega}{\pi\hbar\xi}} \frac{1}{2^l l!} \right]^{1/2} e^{-\theta^2/2} H_l(\theta). \quad (43)$$

The eigenfunction $\phi_l(x)$ is thus expressed purely in terms of bicomplex numbers, the exponential function, and a bicomplex Hermite polynomial.

To the general eigenket (35) corresponds the following eigenfunction:

$$\phi(x) = \left[\frac{m\omega}{\pi\hbar} \right]^{1/4} \left\{ \mathbf{e}_1 \frac{w_{l1} e^{-\theta_1^2/2}}{\sqrt{2^l l!} \sqrt{\xi_1}} H_l(\theta_1) + \mathbf{e}_2 \frac{w_{l2} e^{-\theta_2^2/2}}{\sqrt{2^l (l)!} \sqrt{\xi_2}} H_l(\theta_2) \right\}. \quad (44)$$

The function $\phi(x)$ is normalized, i.e. $(\phi, \phi) = 1$, if

$$|w_{l1}|^2 \mathbf{e}_1 + |w_{l2}|^2 \mathbf{e}_2 = 1. \quad (45)$$

This means that $|w_{l1}| = 1 = |w_{l2}|$. From the properties of real Hermite polynomials, one can show that two functions $\phi(x)$ associated with eigenvalues that differ in both idempotent projections, are orthogonal.

Representation of eigenfunctions

It is instructive to plot some of the functions given in (44). At this stage we do not suggest any specific physical interpretation of the bicomplex eigenfunctions. However, it is useful to see how the standard quantum harmonic oscillator is embedded in the bicomplex harmonic oscillator. In all plots we let $w_{l1} = 1 = w_{l2}$, $\xi_1 = 1$, and we take as independent variable $y = \sqrt{m\omega/\hbar} x$. Dashed lines represent the real part of ϕ while dotted lines represent the hyperbolic part (i.e. the factor of \mathbf{j}). Solid lines represent the function $|\phi|^2$, where $|\cdot|$ is the modulus defined in (11). The normalization factor $(m\omega/\hbar)^{1/4}$ is omitted.

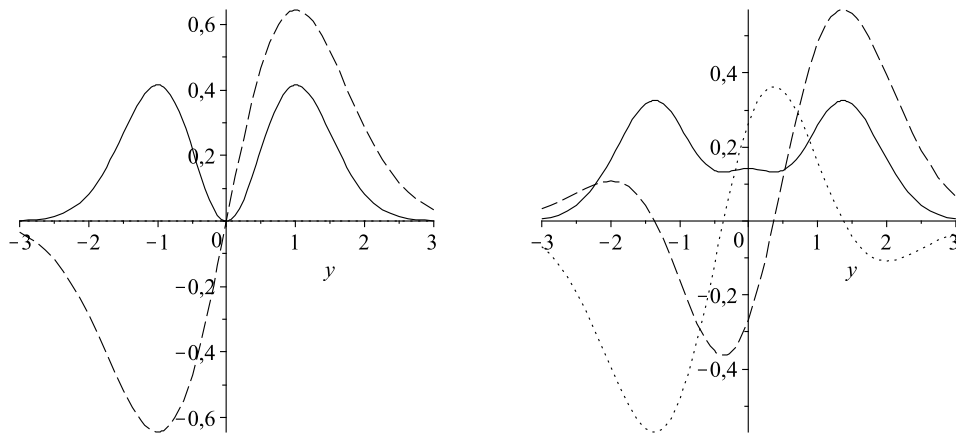


FIGURE 1. Eigenfunction (44) for $w_{l1} = 1 = w_{l'2}$ and $\xi_1 = 1 = \xi_2$. Left: $l = 1 = l'$. Right: $l = 1, l' = 2$. Dashed, dotted, and solid lines represent, respectively, the real part of ϕ , the hyperbolic part of ϕ , and $|\phi|^2$.

In Fig. 1 we let $\xi_2 = 1$. On the left-hand side $l = 1 = l'$. The hyperbolic part of ϕ then vanishes and the real part is equal to the second lowest eigenfunction of the standard harmonic oscillator. On the right-hand side $l = 1$ and $l' = 2$. There is a nonvanishing hyperbolic part in spite of the fact that $\xi = \mathbf{e}_1 \xi_1 + \mathbf{e}_2 \xi_2 = 1$, that is, even if X and P have the usual quantum-mechanical commutation relations.

In all cases where $\xi_1 = 1 = \xi_2$ and $l = l'$, we recover the usual harmonic oscillator eigenfunctions. But these can also be recovered in a different way. One can write $w_{l1} = 1$ and $w_{l'2} = 0$ in (44), in which case the factor of \mathbf{e}_1 coincides with the standard eigenfunction.

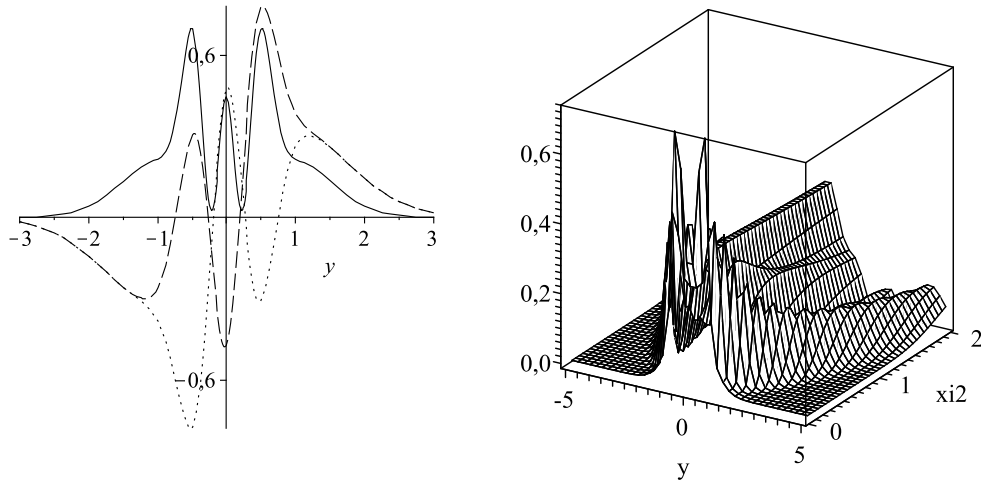


FIGURE 2. Eigenfunction (44) for $w_{l1} = 1 = w_{l'2}$ and $\xi_1 = 1 \neq \xi_2$. Left: $\xi_2 = 0.1, l = 1, l' = 2$, and lines with the same meaning as in Fig. 1. Right: $|\phi|^2$ for $0 < \xi_2 < 2, l = 0, l' = 6$.

Figure 2 displays cases where $\xi_2 \neq \xi_1$, and therefore where the canonical commutation relations are irreducibly bicomplex. On the left-hand side, $\xi_2 = 0.1$, $l = 1$ and $l' = 2$. The right-hand side shows a three-dimensional plot illustrating the variation of $|\phi|^2$ with ξ_2 for fixed ξ_1 , with $l = 0$ and $l' = 6$.

BICOMPLEX HILBERT SPACES

It is well-known [12] that the set of all eigenfunctions of the standard harmonic oscillator Hamiltonian makes up an orthonormal basis of the Hilbert space $L^2(\mathbb{R})$. This property generalizes to the bicomplex harmonic oscillator. The argument is summarized here by means of definitions and theorems, whose proofs can be found in [9].

Definition 1 Let M be a \mathbb{T} -module. For $k = 1, 2$, we define V_k as the set of all elements of the form $\mathbf{e}_k|\psi\rangle$, with $|\psi\rangle \in M$. Succinctly, $V_1 = \mathbf{e}_1M$ and $V_2 = \mathbf{e}_2M$.

Theorem 1 The \mathbb{T} -module M can be viewed as a vector space M' over $\mathbb{C}(\mathbf{i}_1)$, and $M' = V_1 \oplus V_2$.

Definition 2 Let M be a \mathbb{T} -module and let M' be the associated vector space. We say that $\|\cdot\| : M \rightarrow \mathbb{R}$ is a \mathbb{T} -norm on M if the following holds:

1. $\|\cdot\| : M' \rightarrow \mathbb{R}$ is a norm;
2. $\|w \cdot |\psi\rangle\| \leq \sqrt{2}|w| \cdot \| |\psi\rangle \|$, $\forall w \in \mathbb{T}, \forall |\psi\rangle \in M$.

Theorem 2 A bicomplex scalar product induces a \mathbb{T} -norm defined as $\|\cdot\| = |\sqrt{(\cdot, \cdot)}|$.

Definition 3 Let M be a \mathbb{T} -module and let (\cdot, \cdot) be a bicomplex scalar product defined on M . The space $\{M, (\cdot, \cdot)\}$ is called a \mathbb{T} -inner product space.

Definition 4 A *bicomplex Hilbert space* is a \mathbb{T} -inner product space M which is complete with respect to the induced \mathbb{T} -norm.

Theorem 3 Let $\{M, (\cdot, \cdot)\}$ be a bicomplex \mathbb{T} -inner product space, and let the induced space V_k ($k = 1, 2$) be complete with respect to the $\mathbb{C}(\mathbf{i}_1)$ -valued inner product specified as the idempotent projection of (\cdot, \cdot) restricted to V_k . Then $\{M, (\cdot, \cdot)\}$ is a bicomplex Hilbert space.

Definition 5 The *bicomplex harmonic oscillator function space*, denoted by M_{HO} , is defined as the set of all functions $u(x)$, given in (37), where $u_1(x)$ and $u_2(x)$ are standard harmonic oscillator wave functions, that is, elements of $L^2(\mathbb{R})$.

Theorem 4 M_{HO} is a bicomplex Hilbert space.

Theorem 5 Any bicomplex function $u(x)$ in M_{HO} can be expanded uniquely as

$$u(x) = \sum_{l=0}^{\infty} w_l \phi_l(x), \quad (46)$$

where w_l is a bicomplex number and $\phi_l(x)$ is the bicomplex harmonic oscillator eigenfunction given in (43).

CONCLUSION

We have shown that the problem of the quantum harmonic oscillator can be solved in the context of bicomplex numbers, by explicitly obtaining the eigenvalues and eigenfunctions of the bicomplex harmonic oscillator Hamiltonian. It is likely that other quantum-mechanical problems, for instance the hydrogen atom, can be solved in similar ways. The question of whether all postulates of quantum mechanics can be adapted to bicomplex Hilbert spaces is still open and deserves, in our opinion, a careful investigation.

ACKNOWLEDGMENTS

DR is grateful to the Natural Sciences and Engineering Research Council of Canada for financial support. RGL would like to thank the Québec FQRNT Fund for the award of a postgraduate scholarship.

REFERENCES

1. J. von Neumann, *Mathematical Foundations of Quantum Mechanics*, Princeton University Press, Princeton, 1955.
2. G. Birkhoff and J. von Neumann, *Ann. Math.* **37**, 823–843 (1936).
3. S. L. Adler, *Quaternionic Quantum Mechanics and Quantum Fields*, Oxford University Press, Oxford, 1995.
4. G. Baley Price, *An Introduction to Multicomplex Spaces and Functions*, Marcel Dekker, New York, 1991.
5. D. Rochon and S. Tremblay, *Adv. Appl. Clifford Alg.* **14**, 231–248 (2004).
6. D. Rochon and S. Tremblay, *Adv. Appl. Clifford Alg.* **16**, 135–157 (2006).
7. R. Gervais Lavoie, L. Marchildon, and D. Rochon, <http://arxiv.org/abs/1001.1149>.
8. R. Gervais Lavoie, L. Marchildon, and D. Rochon, <http://arxiv.org/abs/1003.1122>.
9. R. Gervais Lavoie, L. Marchildon, and D. Rochon, <http://arxiv.org/abs/1006.5017>.
10. D. Rochon and M. Shapiro, *Analele Universitatii Oradea, Fasc. Matematica* **11**, 71–110 (2004).
11. L. Marchildon, *Quantum Mechanics: From Basic Principles to Numerical Methods and Applications*, Springer, Berlin, 2002.
12. P. Szekeres, *A Course in Modern Mathematical Physics*, Cambridge University Press, Cambridge, 2004.